VOLUME XLVIII NUMBER SEVEN

The Mathematics Teacher

NOVEMBER 1955

A mathematics teacher's creed
J. O. HASSLER

The meaning of mathematics
C. E. SPRINGER

The solution of linear equations and inequalities
W. J. KLIMCZAK

A note on "variable"
ALBERT A. BENNETT

A new mathematical association contest
DANIEL B. LLOYD

The official journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Mathematics Teacher *is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges and Teacher Education Colleges.*

Editor and Chairman of the Editorial Board

H. VAN ENGEN, *Iowa State Teachers College, Cedar Falls, Iowa*

Assistant Editor

I. H. BRUNE, *Iowa State Teachers College, Cedar Falls, Iowa*

Editorial Board

JACKSON B. ADKINS, *Phillips Exeter Academy, Exeter, New Hampshire*
PHILLIP S. JONES, *University of Michigan, Ann Arbor, Michigan*
E. L. LOFLIN, *Southwestern Louisiana Institute, Lafayette, Louisiana*
PHILIP PEAK, *Indiana University, Bloomington, Indiana*
M. F. ROSSKOPF, *Teachers College, Columbia University, New York 27, New York*
HELEN SCHNEIDER, *Oak Junior High School, La Grange, Illinois*

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street N. W., Washington 6, D. C.

Executive Secretary

M. H. AHRENDT, *1201 Sixteenth Street, N. W., Washington 6, D. C.*

Officers for 1954–55 and year term expires

President

MARIE S. WILCOX, *Thomas Carr Howe High School, Indianapolis 7, Indiana, 1956*

Past-President

JOHN R. MAYOR, *University of Wisconsin, Madison 6, Wisconsin, 1956*

Vice-Presidents

CHARLOTTE W. JUNGE, *Wayne University, Detroit, Michigan, 1956*
H. VERNON PRICE, *University of Iowa, Iowa City, Iowa, 1956*
MILTON W. BECKMANN, *University of Nebraska, Lincoln, Nebraska, 1957*
FRANCIS G. LANKFORD, JR., *Longwood College, Farmville, Virginia, 1957*

Board of Directors

HOWARD F. FEHR, *Teachers College, Columbia University, New York 27, New York, 1956*
PHILLIP S. JONES, *University of Michigan, Ann Arbor, Michigan, 1956*
ELIZABETH JEAN ROUDEBUSH, *815 4th Avenue North, Seattle 9, Washington, 1956*
CLIFFORD BELL, *University of California, Los Angeles 24, California, 1957*
WILLIAM A. GAGER, *University of Florida, Gainesville, Florida, 1957*
CATHERINE A. V. LYONS, *University School, Pittsburgh, Pennsylvania, 1957*
JACKSON B. ADKINS, *Phillips Exeter Academy, Exeter, New Hampshire, 1958*
IDA MAY BERNHARD, *Texas Education Agency, Austin 11, Texas, 1958*
HENRY SWAIN, *New Trier Township High School, Winnetka, Illinois, 1958*

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorised March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume XLVIII, *number* 7 November 1955

A mathematics teacher's creed, J. O. HASSLER 450

The meaning of mathematics, C. E. SPRINGER 453

The solution of linear equations and inequalities, W. J. KLIMCZAK 460

A note on "variable," ALBERT A. BENNETT 464

A new mathematical association contest, DANIEL B. LLOYD 469

DEPARTMENTS

Devices for the mathematics classroom, EMIL J. BERGER 473

Historically speaking,— PHILLIP S. JONES 477

Mathematical miscellanea, PAUL C. CLIFFORD *and* ADRIAN STRUYK 481

Mathematics in the junior high school, LUCIEN B. KINNEY *and* DAN T. DAWSON 484

Memorabilia mathematica, WILLIAM L. SCHAAF 488

Reviews and evaluations, RICHARD D. CRUMLEY *and* RODERICK C. MCLENNAN 492

What is going on in your school? JOHN A. BROWN *and* HOUSTON T. KARNES 495

Have you read? 468, 476, 483; *What's new?* 487

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Sixteenth Christmas Meeting 497

Minutes of the Sixth Delegate Assembly 498

Notes from the Washington Office 502

Your professional dates 503

Points and viewpoints 504

THE MATHEMATICS TEACHER is published monthly eight times per year, October through May. The individual subscription price of $3.00 ($1.50 to students) includes membership in the Council. Institutional subscription: $5.00 per year. Single copies: 50 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics,* 1201 Sixteenth Street, N.W., Washington 6, D. C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

A mathematics teacher's creed[1]

J. O. HASSLER, *Emeritus Professor of Mathematics, University of Oklahoma, Norman, Oklahoma.*

At times a teacher should state what he believes and support the statement with sound arguments.

A CREED (derived from the Latin *credo*, "I believe") is usually regarded as a statement of doctrinal beliefs, as in religion. Webster defines it also as "a summary of principles." Today I submit to you a statement of beliefs and principles that I wish might be adopted by every teacher of mathematics in Oklahoma or elsewhere.

CREDO

I believe that mathematics has been and still is one of the most important active forces in the development of our civilization and, next to the English language, the most useful subject matter taught in our schools.

I believe that I should have sufficient knowledge of the various divisions of mathematics, their origins and applications, so that I may correctly interpret the subject as I teach the principles and techniques of mathematical operations, thus being able to fill my students with respect and admiration for such a wonderful discipline.

I further believe that if I am to teach mathematics conscientiously, as its value deserves, I must keep abreast of new developments in the art of teaching it and be alert in my search for its new applications, that arise continually in modern life.

APOLOGIA

In support of the first sentence of the creed let us contemplate what would happen if mathematics were suddenly obliterated from the minds of the human race, all mathematical publications destroyed, every formula in physics, engineering, and commercial practice deleted, every vestige of mathematics erased.

Designs for new airplanes would be halted.

No blueprints for new buildings could exist—in fact, all engineering construction would suddenly cease.

Banks would close. They couldn't compute interest—or even make change.

All commercial activity would suddenly halt—except primitive methods of barter.

Ocean-going ships would lie idly at their piers and transoceanic airplanes would be grounded.

No train schedules could be made.

There would be no method of measuring time other than by sand trickling through a restricted opening or the burning of a knotted rope, knot by knot.

Our activities would be reduced to those of the pre-civilization period of history.

It is horrible to contemplate, but it is true! Imagine being compelled to start all over again from scratch!

Our greatest need—after a language for communication—would be a system of counting, of numbers, with rules of combination. I hope it would not again take so many millenia for someone to think of a zero and give us a positional system, but I suppose we would struggle for centuries upon centuries doing computation on an abacus as we once did long after the golden age of art and literature in Greece. If so, then again civilization would stagnate or mark time, until the invention of

[1] Excerpts from a speech made before the Oklahoma Council of Teachers of Mathematics, October 29, 1954.

sufficient mathematics ushered in once more an age of science and modern civilization.

If all this seems too fantastic, remember my hypothesis is that every bit of mathematics is removed from the minds of men.

We would need a system of measurement—and do we not all hope that next time there would be uniformity in the units invented? From measurement would grow a crude geometry, as it did once before, to blossom into a beautiful pattern of logical demonstration. Perhaps it would again precede the refinement of the number system.

In our benighted condition we could live without history, or geography, or a foreign language, or music, or baton twirling, or football, or the art of glib public speaking, or several other dominant subjects in our present-day curriculum, the promotion of some of which requires much absence from school, participating in inter-school and even state-wide meets. Granted that it would be valuable to study government or economics, or even the facts of geography, in a few years we would find that much of what we had learned had become obsolete, giving way to new economic theories or new geographical boundaries, as we have seen in our generation. But when we would have established that 5 multiplied by 7 equals 35, or that the square on the hypotenuse of a right triangle equals the sum of the squares on the legs (under Euclidean postulates), or that 5 per cent of 200 dollars is 10 dollars (need I mention more examples?) and taught it to the youth, *that* learning would remain permanent, indeed more permanent than the granite hills of South Dakota on which we have carved the images of our national heroes.

Mathematics would probably grow again as it grew before—like a tree. As it was becoming rooted it put out two branches, algebra and geometry, though the algebra branch was no more than arithmetic until later developments. Each put out twigs which grew into supplementary branches, e.g., analytic geometry, non-Euclidean geometry, projective geometry, etc. A third great branch, analysis, grew out, which, starting as calculus very late in the life of the tree, has become the most fruitful part of the tree in its contribution to the progress of civilization. All branches, however, are intertwined in such a way that each is helped much by the others. Mathematics is continually growing in all directions.

Civilization in our modern scientific age lives on the fruit of this tree.

You may conduct a system of barter without any rules of arithmetic, but not a system of banking. You can build a pyramid without trigonometry, if you have the expendable lives of 100,000 slaves at your command, but not a skyscraper. You can build a crude wagon without calculus but not an automobile. You can build a boat without differential equations but not an airplane. Huge airliners that outweigh Columbus' ships now cross the Atlantic in half as many hours as it took him days. Without trigonometry Columbus could not have crossed at all. Without partial differential equations, we would not be flying all over the world now. In both these cases the needed branch of the tree of mathematics had sprouted and grown many, many decades before; and when the need for it arose, it was waiting to be used.

Other branches on that tree are being added every year, some of which are growing only for the pleasure of the research-minded men who create them. But who knows what great development in the civilization of the next century may be sparked by some part of the hundreds of pages of mathematical research now published annually? It has happened before, it will doubtless happen again. And who knows what student in one of your classes may become one of the long line of mathematicians performing this invaluable service to mankind if given the necessary incentive and inspiration by a teacher who knows enough to lead him on?

In justifying the second sentence of the

creed I shall admit at the outset that no one person does or can know all the mathematics that exists today. But one who teaches even the most elementary parts of it can be helped in a very practical way by a knowledge far in advance of what he teaches.

A background of knowledge of analytic geometry will enable him to illuminate the teaching of graphs (not "grafts," as I once heard a teacher say over and over to his class) even in junior high school mathematics. Courses in analysis (calculus) will lead to a clearer understanding of the functional relation so vital in algebra. The attempted display of knowledge by half-informed teachers concerning the concept of infinity and the form $\frac{\text{"0"}}{0}$ has many a time caused confused thinking among students that could have been avoided if the teacher had known calculus. Courses in modern higher geometry, including non-Euclidean and projective, will give a teacher a more wholesome viewpoint of high school plane geometry. Courses in mathematics of finance will show a teacher how the simple algebra of geometric progressions is used in annuity problems, and he can point out how the heralded "5 per cent plan" for financing installment loans is really almost a 10 per cent interest charge. A study of physics enables him to explain the origin of many formulas now used in our texts, for example, "$E=\frac{mv^2}{2}$"; and with this formula he can teach a lesson on why fast driving is dangerous.

In a broader, more general way, a broad horizon in mathematical knowledge is of practical classroom value in that it gives a teacher confidence and enables him to dominate classroom procedure.

What is more pitiful than the hesitating, uncertain teacher, who, without any background of wide experience in higher mathematics to give him a commanding grasp of his elementary subject, must refer to his answer book, or try vainly to recall how the exercise was worked out in the key he studied in preparation for the lesson? As his weakness is revealed to his pupils, they look on him with disdain and enjoy "stumping" him with hard problems. Consider, on the other hand, the teacher who "knows more than is in the book"; who does not blindly teach the errors in the text, but points them out to the pupils with such a clear, understandable explanation that the pupils admit the error and recognize his transcendent knowledge of things mathematical; who can lead his best pupils into higher realms of mathematical thought; and who thereby wins the approbation and good will of the class. Sometimes this helps more with discipline than a knowledge of psychology. It was said of the great penman, Spencer, that his class sat in rapt attention when he stepped to the board to set a copy, because of his consummate skill.

Many people make a serious mistake in thinking higher mathematics is useful only in engineering and physics. At a research conference on economics one paper presented was titled "The Relation of Stock Prices to Earnings and Other Factors." The economist author referred to and used linear differential equations. Partial differential equations are used in research on genetics.

Within a decade (almost) there has arisen a mathematical theory of games of strategy. Checkers, chess, the *strategy* of war and what not are being put on a mathematical basis. Perhaps even love and courtship will come next. After all, are they not sometimes games of strategy?

There is no need for an apology for the third sentence of the creed. Any teacher who cannot subscribe to that should consider thoughtfully whether or not he has chosen the right career.

The meaning of mathematics[1]

C. E. SPRINGER, *University of Oklahoma, Norman, Oklahoma.*

The spirit of mathematics is often violated by designating it as a "tool subject," thereby covering up a concept which pervades modern mathematics—the postulational method.

THERE IS PROBABLY no other field so misunderstood as mathematics. That is, there is no other field in which the aims, methods, and philosophy are so little appreciated by those outside the field as well as by many in the field itself. To the mathematician's next door neighbor, those hours of mathematical research carried on in spare time appear to be directed toward finding short cuts for that unpleasant job which confronts us just before April 15 each year. The concept of mathematics which most laymen hold is that it comprises nothing more than arithmetic. All too often it is true that, to both student and teacher of mathematics, their concept of the meaning of mathematics is derived from the traditional courses as far as they have progressed—let us say classical elementary algebra, in which we never become sufficiently brave to let x be anything but some *number*, so-called college algebra, trigonometry with its time-consuming solutions of triangles, analytics, calculus, plus perhaps differential equations with its bag full of tricks and rule-of-thumb procedures, solid analytics, beginning statistics, mathematics of finance, mechanics, theory of equations, and perhaps further courses—all taught in their watertight compartments. One may make high marks in all of these courses without becoming aware of the real significance of mathematics. To the scientist, mathematics is a sister science which was developed as an adjunct to his particular field—a set of tools for those skilled enough to use them.

To the professional mathematician, mathematics is all of these things and more. Actually, many professional mathematicians regard their subject as closely related to the fine arts. During the course of its development, mathematics has been all the things ascribed to it by the layman, scientist, and mathematician.

I wish to point out some characteristics of modern mathematics of which all teachers of mathematics should be aware. It is extremely important for the development of science and mathematics and, therefore, for the future of our country that more people gain an understanding of the nature of modern mathematics. It is certainly desirable that mathematics students and teachers be educated against thinking of mathematics as mere pencil pushing. Besides teachers and students of mathematics, it is possible for intelligent laymen to gain something of the spirit of modern mathematics without having to trudge through the thick underbrush of technical detail which makes up so large a part of some of our elementary college courses. Some useful books for both the layman and the mathematics teacher are listed in the references.

Through the centuries the term "mathematics" has become associated with a vast accumulation of techniques and processes. And, no doubt, mathematicians of the past like the professional mathematicians of today have been somewhat annoyed at times to hear some layman

[1] A paper read at the Mathematics Teachers Institute at the University of Oklahoma, June 9, 1954.

give a wrong impression of what the mathematical researcher does. If you have read that charming little book by the late Professor G. H. Hardy entitled, *A Mathematician's Apology*,[2] you will recall how bitterly Hardy attacked Lancelot Hogben, author of *Mathematics for the Million*.[3] Hogben's crime, according to Hardy, was to give the impression to his millions of readers that the elementary mathematics (which was all he knew) was all of mathematics. Hardy classifies mathematics under two headings: "school" mathematics and "real" mathematics. The former would go as far as the elementary calculus, and the latter would involve modern abstract mathematical systems.

From the beginning, mathematics has played a triple role: the mystic, the practical, and the pure mental discipline. At the earliest, mathematics was close to religion. Astrologers and numerologists were plentiful. The mystic aspect of mathematics still gains more than its share of publicity. This may be an interesting subject to contemplate, but it is certainly only of historical interest to mathematicians and not an important facet of the field from our modern point of view, so we say no more about it.

It cannot be denied that the use of mathematics as a technique has been and still is indispensable in the development of our civilization. I selected the word, "technique," here rather than "tool" advisedly, because I believe that the term, "tool subject," as applied to arithmetic, gives the impression that arithmetic has only one value, namely, the learning of the trivial rules of addition, subtraction, and the rest. Experiences of a wholly different order may be had from the study of numbers. I believe with Charles H. Judd who wrote in the *Third Yearbook* of The National Council of Teachers of Mathematics on "The Fallacy of Treating School Subjects as 'Tool Subjects' " as follows: "The curriculum maker who thinks that he has exhausted the catalogue of uses of number when he has listed the examples which ordinary men solve in a day or a week is superficial to such an extreme degree that he is an unsafe guide in arranging the plans of the school. The man who calls arithmetic a 'tool subject' and with this name dismisses it as something less worthy than subject-matter courses is guilty of criminal neglect of true values."[4]

If I describe mathematics as a convenient tool for the physicist or engineer, I do not mean the narrow use of the word "tool" which Judd described in his article. The high level to which applied mathematics has risen in our time is a source of amazement even to mathematicians. Perhaps the best way for one to become adept as an applied mathematician now is to become first somewhat of a pure mathematician. This means that mathematics has now become a profession. Besides in the profession of teaching mathematics, a mathematician may now serve in hundreds of ways in governmental positions, in industrial positions, and in research organizations. Mathematics is now applied at almost all levels. It may be elementary algebra, actuarial mathematics, the calculus and Fourier methods in electrical engineering, Boolean algebra in a Naval project, matrices, or tensor analysis in an applied physics problem.

From the time of the early Egyptians, mathematics was useful to the surveyor, the astronomer, the engineer, the physicist, the medical researcher, the statistician, any many others. Duncan Harkin says in his very useful and interesting book entitled *Fundamental Mathematics*, "to build with line and number instead of sticks and stones, to put them together with plus and times instead of nail and mortar, to make an ideal structure in

[2] G. H. Hardy, *A Mathematician's Apology* (Cambridge, England: The University Press, 1940), p. 77.

[3] Lancelot Hogben, *Mathematics for the Million* (New York: W. W. Norton and Company, 1937).

[4] Charles H. Judd, "The Fallacy of Treating School Subjects as 'Tool Subjects'," *Third Yearbook*, The National Council of Teachers of Mathematics, Washington, D. C., 1928, Chapter I, pp. 1–11

place of a tangible toy—this is mathematics."[5] I am afraid we should reprimand Mr. Harkin for using the phrase: "this is mathematics." This is part of mathematics—a very important part—the tool aspect again, or applied mathematics. It cannot be stressed too strongly that many if not most of the important phases of applied mathematics came about as mere residues of the development of mathematics as a pure mental discipline. But I should like to pick up Harkin again. He continues and does indicate the role of mathematics as a mental discipline. "This [mathematics] is one of the houses our mind can live in: to some a simple shelter from the real world outside: to others, a work shop where tools are fashioned to gain dominion over that outside world, to tunnel the earth, to make the air yield to flight, to make fire work, to make steam turn mechanical water wheels and electrical turbines. Within the shelter we find beauty of design, symmetry of development, and fascinating connections between the parts. In the work shop we learn to be architect, engineer, physicist, scientist."[6]

Yes, it is foolish to deny the obvious fact that mathematics has marked our lives directly or indirectly by having been extremely important in developing such amenities of modern life as elevators, airplanes, and radios. Fortunately or unfortunately it has also been responsible for a large part of the development of such things as rockets, atom bombs, and television! In short, mathematics is a *sine qua non* for much of modern science and technology. And it is upon mathematics that much hope is placed for even further development of our modern technology. But I must leave the workshop and go back into the shelter.

In the shelter is housed the third role of mathematics, that of the mental discipline. Of the many compartments in the shelter, probably the largest and most important room is that where we expect to find out something significant about the title of this paper, "The Meaning of Mathematics." Before we have a look at the product from this room, we might investigate a little the evolution of the meaning of mathematics. We might find it illuminating to try to find out how mathematics got to where it now is. That is, it may be useful to gain perspective for the characterization of modern mathematics by reviewing briefly some characterizations which have been outgrown and outmoded, or, perhaps more properly said, subsumed.

At one time mathematics could rightly be described as the science of number, but this is far from valid today. From the situation of the savage who had words for only one, two, three, and many, to the invention of a symbol to represent zero was a long period of development of mathematics and also of civilization.

Another popular definition of mathematics is: "mathematics is the science of measurement." After the primitive mathematicians evolved a number system, i.e., after they had put certain grunts into one-to-one correspondence with the notions of what we call one, two, three, etc., the next question attacked was "how long?" "how far?" "how tall?"

In answering the question "how far?" the mathematician of ancient times reduced the problem to that of "how many?" It was a journey of so many days, or to use an expression of our own pioneers, it was, say, three wagon greasings from Tennessee to Oklahoma. A horse is 15 hands high, etc. As late as the time of the philosopher August Comte, mathematics was thought (by some who should have known better) to be the science of measurement. The theorem in Euclidean geometry, that the sum of the angles of a triangle is two right angles, cannot be proved by the use of the most precise measuring gadgets known to man. It is proved only by logical deduction from a

[5] Duncan Harkin, *Fundamental Mathematics* (New York: Prentice-Hall, 1941), p. vii.
[6] *Ibid*, p. ix.

set of postulates. So mathematics is not entirely the theory of measurement.

As geometry began to develop, it was included in the domain of mathematics—so mathematics was defined as the science of space. Probably a certain amount of abstraction took place relative to the sticks with which one made figures—in both two and three dimensions—so that the concept of a line segment was born. Then came the idea of constructing figures in the imagination, and in the third century B.C. the deep thinkers could show that many of the facts concerning the configurations of sticks could be deduced logically from a few facts concerning them. Thus, logic became a substitute for experiment in geometry, and Euclid was able to write a compendium of knowledge which is still in use. By modern standards there were loose arguments and appeals to the intuition in much of Euclid's work. But an acceptable presentation by modern standards was not arrived at until about 1904. A great advance in intellectual progress was made by Euclid, because he set the pattern with his axiomatic method for much of modern development in mathematics. He, of course, was not able to look as carefully into his thought processes as we are today.

Today, we investigate properties of various kinds of spaces, and we abstract from and generalize various facets of Euclidean geometry, but mathematics now means much more than a proliferation of geometries.

We come to the question: "What is mathematics?" If one consults *Webster's Collegiate Dictionary*, one finds an unenlightening statement. To one who knows little about our field a satisfactory definition can be given by some such statement as this: "Mathematics is the deductive science which investigates spacial and numerical relations." Now the investigation of spacial and numerical relations is in the domain of mathematics, but such a small segment of the domain that this is not only a useless definition but a misleading one.

Some great mathematicians, being dissatisfied with such unsatisfactory dictionary definitions, have attempted, again rather unsatisfactorily, to embrace all of mathematics in one brief phrase. Benjamin Pierce said, "Mathematics is the science which draws necessary conclusions." Bertrand Russell said, "Mathematics is the set of all propositions of the form P implies Q." These are apt statements of characteristics of the subject, but they are too technical and too esoteric to mean much unless one is already sure of his ground in the field. You have no doubt heard another famous dictum of Russell, namely, "Mathematics is the science in which we never know what we are saying nor whether what we are saying is true!" Now this is, of course, a very clever statement about mathematics, but it is for consumption and delectation of mathematicians who already know the subject to some advanced level. If one appreciates what Russell meant by this statement, then he has fairly good insight into the meaning of mathematics. Somewhat of an improvement on Russell's statement (at least for those who might misunderstand Russell's attempt) is the following offered by C. J. Keyser of Columbia University: "Sheer mathematics is the science in which one never thinks of a definite sort of subject matter, nor fails to know that what one asserts is true." We shall try to make this more meaningful later.

A very fine book entitled *What is Mathematics?* has been written by Courant and Robbins.[7] They do not attempt to define mathematics in a simple sentence, but they give some wonderfully elucidating material in about 500 pages. Their feeling is that in order to know what mathematics is, one must know some mathematics. They remark that "understanding of mathematics cannot be transmitted by painless entertainment any more than education in music can be brought by the

[7] Richard Courant and Herbert Robbins, *What Is Mathematics?* (London: Oxford University Press, 1941).

most brilliant journalism to those who never have listened intensively."[8] Courant gives some comments on the nature of mathematics which are worthy of contemplation. "Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection."[9] And note the apparent antithetic expressions in the following quotation from Courant. "Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science."[10]

David Hilbert (with Cohn-Vossen) wrote a book in German, entitled *Anschauliche Geometrie*, which appeared recently in English translation under the title *Geometry and the Imagination*.[11] This is a book which should be in every mathematics library. Some of Hilbert's statements in the preface are of interest in a discussion of the meaning of mathematics. Hilbert wrote, "In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies; a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations." In geometry, for instance, the abstract tendency has given rise to the magnificent systematic theories of algebraic geometry, of Riemannian geometry, and of topology, which theories employ abstract reasoning and much symbolic calculation. However, it is still true of these abstract branches of geometry that intuitive understanding plays an important role, not only as an aid in research but also as a means of breaking through the abstract barrier to a better comprehension of the situations involved. By means of the light of visual imagination the manifold facts, problems, and concepts of geometry are illuminated, and further, it is often possible to depict the geometric outline of the methods of investigation and proof, even though the visual imagination is not enough to appreciate the fine details of the logic and the actual calculations. For instance, many results of combinatorial topology can be revealed by using visual imagination, even though the whole subject is strongly interlaced with abstract algebra and group theory. It is the opinion of the authors, Hilbert and Cohn-Vossen, that because geometry, being many-faceted and related to many and diverse branches of mathematics, can be approached through visual imagination, and thus a fairly thorough summarizing survey of mathematics as a whole is available to a wider range of people than just the specialists. This book, *Geometry and the Imagination*, attempts to give the intelligent layman some insight into the intriguing concepts of advanced modern mathematics, and to dispel the impression that modern mathematics is mere juggling of numbers.

In lieu of a definition of mathematics in one sentence, we shall be content to enunciate and elucidate a definition of a mathematical system which is really the core of modern mathematics. A mathematical system is any body of propositions arranged as a sequence of logical deductions; i.e., an ordered set of propositions arranged so that each proposition *after a certain one* is a logical consequence of one or more propositions which precede it in the set. Since there is no statement preceding the first proposition from which it can be deduced, we see that it is necessary to have one or more statements which are

[8] *Ibid.*, p. v.
[9] *Ibid.*, p. xv.
[10] *Ibid.*
[11] David Hilbert and Cohn-Vossen, *Mathematics and the Imagination*, Translated by P. Nemenyi (New York: Chelsea Publishing Company, 1952).

accepted without proof and from which further propositions may be deduced. These propositions which are assumed at the outset are called axioms or postulates. We take the terms, "axiom" and "postulate," as synonymous. The use of the term "postulate" is to be preferred because of the common use of axiomatic as meaning self-evident. It must be strongly impressed upon the student's mind that a postulate is merely an assumption, and not a "self-evident" truth, as some of the older writers would have us believe. The set of postulates which we choose to begin with is entirely of our own choosing. Of course, the postulates must be consistent, and for aesthetic reasons it is desirable that they be independent. The nature of the mathematical system which we obtain on applying some form of logic to the postulates depends upon the choice of postulates. If we choose the proper set, we get as a consequence the Euclidean geometry. Although as I have already indicated, it requires rather advanced thinking to give a proper postulational treatment of Euclidean geometry which is devoid of the use of intuition. If we alter the famous fifth postulate (the parallel postulate of Euclidean geometry), we obtain a non-Euclidean geometry. Non-Euclidean geometry was known to be a consistent mathematical system in the time of Kant, but he apparently did not know about it, or refused to accept it. Bolyai (1802–1860) and Lobatchewsky (1793–1856) were mischievous enough to challenge the fifth postulate of Euclid, and each obtained a new kind of geometry. These geometries have done much to press mathematics, in general, on to its present glory.

The abstract mathematical system which Euclid used was applied to geometrical configurations, and the result is Euclidean geometry. Of course, mathematics was not far enough advanced in the third century B.C. for Euclid to realize that he could substitute nameless marks x, y, z for points, lines, planes, and thus arrive at a beautiful abstract mathematical system—a doctrinal function. The doctrine is Euclidean geometry if the objects x, y, z are points, lines, planes. The fact that the doctrine is a function of the set of postulates chosen is what Benjamin Pierce had in mind when he said that mathematics is the science which draws necessary conclusions. If P is the set of postulates and Q is the set of consequences or theorems which are implied by applying a logic to P, then we see why Russell said that mathematics is the science in which P implies Q.

The great success of the postulational method as applied to geometry was, unfortunately, not carried over to other branches of mathematics. The traditional material of algebra, trigonometry, analytics, and calculus advanced in volume by means of an admixture of logic and intuition without any attempt to reduce the fundamentals to a simple set of postulates. By the middle of the nineteenth century, however, some mathematicians became dissatisfied with the confusion caused by appeals to the intuition, and they began a program designed to eliminate the intuition. The success attending their efforts gave impetus to the attempt to axiomatize all of mathematics. To accomplish independence of intuition, all entities to be studied in a branch of mathematics were to be considered as purely abstract. Mathematics reduces to a study of axiomatics, that is, a study of the implications of a set of postulates. There is nothing in the body of theory resulting which was not implied by the set of postulates. Whether the postulates are in correspondence with some physical situation or not is of no real concern to the pure mathematician. The truth of his results means only: are they logical deductions from the accepted postulates? This may throw some light upon Russell's statement that mathematics is the science in which we never know what we are talking about, nor whether what we are saying is true. It is surprising even to mathe-

maticians that after building an abstract system, it may find an important use in describing and predicting results in connection with some physical phenomenon. A case in point is the application of Boolean algebra to electric circuits used in modern computing machines. Another example of the use of an abstract mathematical machine to describe a physical system is that of tensor analysis in general relativity theory.

The concept of the postulational treatment pervades modern mathematics. Space is not available to go into extensive examples, but I should like to mention a source book of interest. It is *The Anatomy of Mathematics* by Kershner and Wilcox.[12] In this book the materials of mathematics are described, and discourses are set forth about conceptual entities known as sets, relations, functions, operations, and the like. The postulational point of view is dominant throughout. Such mathematical systems as groups and fields are discussed. By the time one has finished this book, he should have a clear idea of the nature and meaning of some mathematical systems and theories, and, therefore, of the postulational procedure used in other mathematical branches of learning.

[12] R. B. Kershner and L. R. Wilcox, *The Anatomy of Mathematics* (New York: The Ronald Press Company, 1950).

Words! Words! Words!

What is a parameter? While this word is not common in the high school curricula, it is one that occurs frequently in the freshman courses in college. It is just another good illustration of how words are sometimes "piled" into "sentences." The false notions developed by such "piling" of words can sometimes cause considerable difficulty for the learner. For example, it comes with considerable surprise to some students of the calculus that they can differentiate with respect to a constant. Isn't the derivative of a constant always equal to zero?

In some books we read that "a parameter is an adjustable constant. It has some of the characteristics of a variable and some of the characteristics of a constant." This is indeed a revealing statement, particularly, when one finds that "A symbol which, throughout a discussion, does not change in value is called a constant." In other instances one finds degrees of being constant. Thus, "*Constants*, as the name signifies, are symbols which have the same value throughout any discussion or problem. Those that have the same definite fixed value at all times are called *absolute constants*." "Constants which represent any arbitrary fixed value . . . are called *arbitrary constants* or parameters." Evidently a parameter is some kind of a constant.

In the equation $y=2x+b$ there are three variables, namely, y, x, and b. If this equation were graphed in three dimensions, it would be a plane. At times it seems desirable to study the intersections of two planes, such as:

A. $y=2x+b$	B. $y=2x+b$	C. $y=2x+b$
$b=1$	$b=-4.5$	$b=0$

Usually this approach is considered to be too difficult for college freshmen, so the idea of a parameter is introduced. But to call it an "arbitrary constant" or a "variable constant," can be confusing to the learner.

A parameter is a variable used in a special way. To each value of the variable (parameter) b in the equation $y=2x+b$, it may be desirable to study the set of all number pairs (x, y) which may be associated with a particular value of the parameter. Such an approach produces families of straight lines (in this instance) if plotted in two dimensions.

With the proper concept of a parameter in mind the student of the calculus should not be too shocked when he is asked to find $\frac{\delta f}{\delta c}$ if $F(x, y, c)=0$.—*Henry Van Engen*

The solution of linear equations and inequalities

W. J. KLIMCZAK, *Trinity College, Hartford, Connecticut.*

An interesting interpretation of the problem of finding an algebraic solution to two consistent linear equations in two unknowns, and the application of the method to linear inequalities.

THE USUAL METHODS of solving a consistent system of linear equations in two unknowns,

$$(1a, b) \quad l_1(x, y) \equiv a_1x+b_1y-c_1=0,$$
$$l_2(x, y) \equiv a_2x+b_2y-c_2=0,$$

consists of reducing $(1a, b)$ to the simpler system,

$$(2a, b) \qquad x=d_1, \quad y=d_2,$$

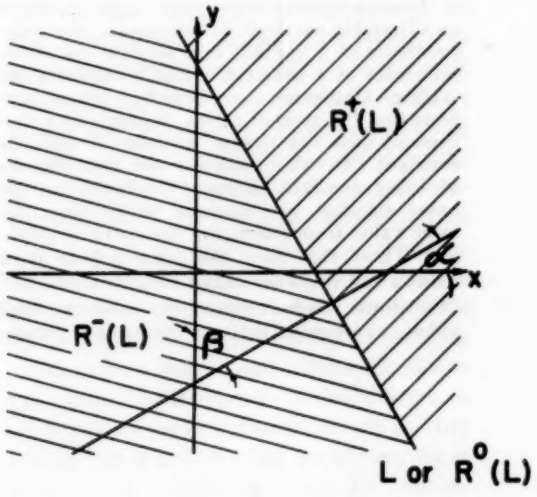
by a sequence of algebraic operations. Thus the process of finding the solution of $(1a, b)$ is essentially one of simultaneously converting the linear forms $l_1(x, y)$ and $l_2(x, y)$ into linear forms of the type $ax+b$ and $cy+d$ respectively. Interpreted geometrically, this algebraic process consists of transforming the lines l_1 and l_2 which have $(1a)$ and $(1b)$ as their equations into lines L_1 and L_2 which are parallel to the y-axis and x-axis respectively and which intersect in the same point as l_1 and l_2. The equations of L_1 and L_2 are given by $(2a)$ and $(2b)$. The purpose of this paper is to present a method of performing this geometric transformation of l_1 and l_2. The method itself may be of interest to both teachers and students of analytic geometry as an application of linear transformations of the xy-plane to the solution of linear equations. Also it has application to the solution of the corresponding system of linear inequalities,

$$(3) \qquad l_i(x, y)<0 \quad (i=1, 2),$$

which will be examined.

We begin by considering any linear form $L(x, y) \equiv Ax+By+C$ with $A \neq 0$. Let L denote the line which has $L(x, y)=0$ as its equation. Now L divides all the points in the xy-plane into three classes which we shall denote by $R^0(L)$, $R^+(L)$, and $R^-(L)$. A point (x, y) is in either $R^0(L)$, $R^+(L)$, or $R^-(L)$, according as $L(x, y)$ is zero, positive or negative. Furthermore, let $K=e(A^2+B^2)^{1/2}$, where $e=+1$ or -1 so that $eB>0$ if $B \neq 0$ and $eA>0$ if $B=0$. Then if α and β denote the direction angles of any line perpendicular to L (see Fig. 1), it follows from the normal form of

Figure 1



the equation of L that $\cos \alpha = A/K$ and $\cos \beta = B/K$.

The following two linear transformations of the xy-plane corresponding to the linear form $L(x, y)$ will be needed. The first of these is given by the equations

$$(4) \qquad x' = x + y \tan \alpha, \quad y' = y.$$

Such a transformation is called a shear and takes each point (x, y) of the plane into a new point with coordinates (x', y') by moving the point (x, y) along a line parallel to the x-axis through the directed distance $y \tan \alpha$. We note that under this transformation the points on the x-axis remain fixed. Thus the line L is transformed into the line L' which has the same x-intercept as L and is parallel to the y-axis. Similarly, if $B \neq 0$, we define a second shear by means of the equations

$$(5) \qquad x'' = x, \quad y'' = x \tan \beta + y.$$

This transformation moves each point (x, y) of the plane along a line parallel to the y-axis through the directed distance $x \tan \beta$, and takes the line L into the line L'', which has the same y-intercept as L and is parallel to the x-axis. In this case the points on the y-axis remain fixed.

We shall now consider the question of simultaneously reducing $l_1(x, y)$ and $l_2(x, y)$ to linear forms of the type $ax+b$ and $cy+d$ respectively by means of suitably defined shears. Since the system $(1a, b)$ is consistent, the determinant of the system $a_1b_2 - a_2b_1$ is not zero, and there is no loss of generality in assuming that $a_1 \neq 0$. We then can define the shear (4) corresponding to $l_1(x, y)$, where $\tan \alpha = b_1/a_1$. Solving (4) for x and y gives

$$(6) \qquad x = x' - y' \tan \alpha, \quad y = y'.$$

By direct substitution it is easy to verify that this transformation reduces $l_1(x, y)$ and $l_2(x, y)$ respectively to

$$(7) \quad l_1'(x', y') \equiv a_1x' - c_1,$$
$$l_2'(x', y') \equiv a_2x' + (a_1b_2 - a_2b_1)y'/a_1 - c_2.$$

Now since the coefficient of y' is not zero, it is possible to define the shear (5) corresponding to $l_2'(x', y')$ by letting $\tan \beta = a_1a_2/(a_1b_2 - a_2b_1)$ and replacing x and y by x' and y' respectively. Solving the resulting equations for x' and y', we have

$$(8) \qquad x' = x'', \quad y' = -x'' \tan \beta + y''.$$

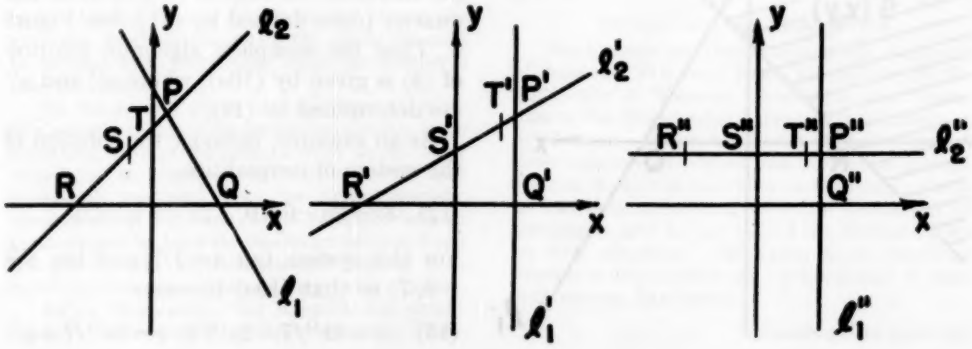
Substituting these equations in (7) reduces $l_1'(x', y')$ and $l_2'(x', y')$ to

$$(9) \quad l_1''(x'', y'') \equiv a_1x'' - c_1,$$
$$l_2''(x'', y'') \equiv (a_1b_2 - a_2b_1)y''/a_1 - c_2$$

respectively. Thus the linear forms $l_1(x, y)$ and $l_2(x, y)$ have been simultaneously reduced to the required forms $l_1''(x'', y'')$ and $l_2''(x'', y'')$ by means of the transformations (6), defined for $l_1(x, y)$, and (8), defined for $l_2'(x', y')$. These two shears can be combined into a single transformation by substituting (8) into (6) to give

$$(10a) \quad x = x''(1 + \tan \alpha \tan \beta) - y'' \tan \alpha,$$

Figure 2



$$y = -x'' \tan \beta + y'',$$

or solving for x'' and y'',

(10b) $$x'' = x + y \tan \alpha,$$

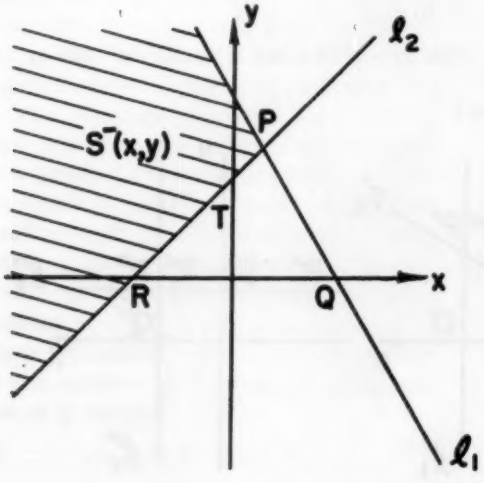
$$y'' = x \tan \beta + y(1 + \tan \alpha \tan \beta),$$

where $\tan \alpha = b_1/a_1$ and $\tan \beta = a_1a_2/(a_1b_2 - a_2b_1)$. Now (10b) can be interpreted geometrically as transforming each point (x, y) of the plane into a point (x'', y'') so that l_1 and l_2 are transformed into lines l_1'' and l_2'' which are parallel to the y-axis and x-axis respectively. Figure 2 illustrates this transformation.

If we let $S^0(x, y)$ denote the set of all points (x, y) which are simultaneously in the sets $R^0(l_1)$ and $R^0(l_2)$, then it is clear that the solution of (1a, b) consists of all points in $S^0(x, y)$. Now $S^0(x, y)$ contains a single point which is transformed by (10b) into the point (x'', y'') whose coordinates can be obtained from (9) by equating $l_1''(x'', y'')$ and $l_2''(x'', y'')$ to zero. Thus $x'' = c_1/a_1$ and $y'' = a_1c_2/(a_1b_2 - a_2b_1)$. Substituting these values into (10a) gives $x = (c_1b_2 - c_2b_1)/(a_1b_2 - a_2b_1)$ and $y = (a_1c_2 - a_2c_1)/(a_1b_2 - a_2b_1)$ as the solution of (1a, b).

We shall now derive a similar method for solving the system of linear inequalities (3). The usual method of solving such a system found in textbooks on higher algebra is geometric. The solution, which we shall denote by $S^-(x, y)$, consists of a shaded region of the xy-plane which contains all points (x, y) which are simultaneously in $R^-(l_1)$ and $R^-(l_2)$. See Figure 3. However, no purely algebraic method is given corresponding to that used to solve the system of linear equations (1a, b), which reduces the system to a set of equations each involving one unknown. The algebraic solution of (3) can be obtained by applying the transformation (10a).

Figure 3



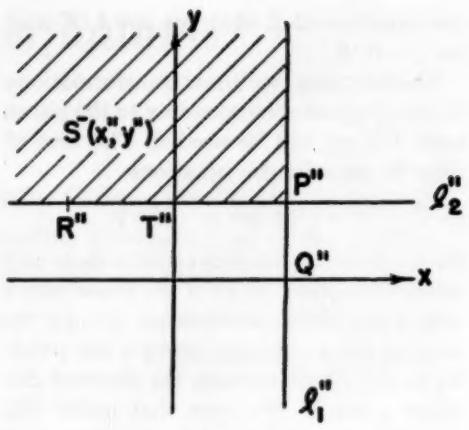


Figure 4

We shall let $S^-(x'', y'')$ denote the set of points into which $S^-(x, y)$ is transformed by (10a). Now when (10a) is applied to the xy-plane, it follows from (9) that (3) is reduced to the simpler system of inequalities,

(11) $$a_1x'' < c_1, \quad (a_1b_2 - a_2b_1)y''/a_1 < c_2,$$

which has the set of points $S^-(x'', y'')$ as its solution. Clearly $S^-(x'', y'')$ is the quarter plane defined by (11). See Figure 4. Thus the complete algebraic solution of (3) is given by (10a), where x'' and y'' are determined by (11).

As an example, consider the solution of the system of inequalities,

(12) $$3x + 2y - 6 < 0, \quad 2x - y + 7 < 0.$$

For this system $\tan \alpha = 2/3$ and $\tan \beta = -6/7$, so that (10a) becomes

(13) $$x = 3x''/7 - 2y''/3, \quad y = 6x''/7 + y''.$$

The transformation (13) reduces (12) to the simpler system,

(14) $x'' < 2, \quad y'' > 3.$

The solution of (12) is then defined by (13), where x'' and y'' are any real numbers which satisfy (14).

United States complacency hit, Soviet mathematics hailed

With the recent appointment to the Atomic Energy Commission of Dr. John von Neumann, one of the leading mathematicians in the United States, and with the consequent heightening of interest in the position of higher mathematics in the field of nuclear physics, it is more than pertinent today to study the role and status of mathematicians in the Soviet Union.

Although the Kremlin continues to wrap and rewrap in secrecy many of the developments in Soviet mathematics, many significant facts are available.

Pieced together, the facts present a disturbing picture. They speak against any complacency Americans might happen to have about their own leadership in either "pure" or "applied" mathematics.

The leading conclusion of most Western scholars themselves is that "native" Soviet mathematicians are among the best in the world —and, in some respects, the best.

Soviet mathematicians working in the field of differential equations, for example, are reputed to constitute the most outstanding group of such specialists to be found in any nation, free or Communist. In general, it is acknowledged that, while the American school of mathematicians still holds the lead, the Soviet school is so close in comparison that it serves as an extremely stiff—and actually the only—competitor.

Soviets honor mathematics

Two other conclusions should be emphasized. In the first place, mathematics as a discipline is much more highly honored in Russia than it is in the United States. In the Soviet Union, for example, mathematicians on the whole are said to outrank even physicists, while the reverse is true in America.

In the second place, the directors of Moscow's natural scientific institutes devote more attention to mathematicians who apply their knowledge to practical, natural, scientific, and industrial problems than to abstractionists. Mathematical logicians, to be sure, are held in great respect by both the Soviet government and the public, but mathematicians dealing with more concrete questions receive greater support.

As an illustration, the Kremlin has established the Central Aero-Hydrodynamic Institute, an academy devoted to nothing else but the pursuit of such specific problems as those connected with fluid mechanics and the theory of oscillations. There scores of specialists are engaged in research costing millions of rubles. In the United States relatively few men are involved in such work, and their activities cost only pennies by comparison.

Experts listed

There are dozens of Soviet mathematicians who are worthy of special mention, but the following five can be taken as typical of the present-day Soviet masters in this field.

First, there is A. A. Kolmogorov, the so-called "Führer" of the Soviet school, one of the top three or four mathematicians in the world today. In spite of the fact that Mr. Kolmogorov bears the theoretical stigma of coming from a family of well-to-do Moscow merchants, he is a member of the Communist Party, a member of the Academy of Sciences, and for many years has been director of Moscow University's Institute of Mathematics. As in the case of almost all of the Soviet Union's outstanding mathematicians at the present time, Kolmogorov is comparatively young. He is still in his forties.

Mr. Kolmogorov's forte is research connected with the theory of probabilities, so important in the development of any atomic energy program, peaceful or warlike. In this respect, he is the Soviet counterpart of Dr. von Neumann who is noted here for his pioneer work in electronic computing devices, thereby greatly contributing to the development of the hydrogen bomb.

United States visited twice

Next comes another leader in the Academy of Sciences, Professor P. S. Alexandrov of the University of Moscow. Older than most members of the Soviet school, Professor Alexandrov, by the same token, has had a chance to travel more extensively beyond the borders of Russia. He has made regular visits to Germany, where he has been associated with the University of Göttingen, and he has visited the United States on two occasions. Colleagues have described Professor Alexandrov as "tyrannical," a man with strong, fixed ideas.

Continued on page 472

A note on "variable"

ALBERT A. BENNETT, *Brown University, Providence, Rhode Island.*

It is important that teachers give serious thought to the meaning of the basic terms used in their classes. This note will help to clarify the term "variable."

THE VOCABULARY of mathematics is rich in terms suggesting change and variation. "Proportion and variation" is a traditional topic in elementary algebra, as is "related rates" in calculus; and it is well known that differential equations are ideally fitted to handle certain laws of varying physical action. But variability is not merely, like finance, a domain for the application of mathematics. A mathematical function may be increasing; the terms of a series may approach zero. Limits are often described in using the phrase "becomes and remains." Basic topics in modern mathematical analysis lie in the theories of functions of real and of complex variables respectively. One speaks of the rate of growth of a function. Newton used "fluxions" to name the topic that we usually call "calculus." Modern algebra may start with linear transformations, and projective geometry with projective transformations. Covariants no less than invariants are sought for in the theory of quantics. Topology treats of properties preserved under continuous deformations. A discontinuous function may have a "jump" at a point of discontinuity. Sometimes one must choose arbitrarily some one member of a given class. Though the chosen element is fixed, the manner of selection may well result in different choices in different instances.

The function concept has been much in the fore during the last few decades, ever since E. H. Moore, of the University of Chicago, urged that elementary mathematics should concern itself more with the grasp of far-reaching unifying ideas (such as that of function) and less with unmotivated and uncomprehended drill. Usually functions are explained in terms of independent and dependent variable, so that "variable" (which may be studied without the notion of mathematical functions) becomes of prior importance.

It seems eminently fitting that teachers of mathematics should gain some understanding of the concept "variable" and of its use in mathematics. As is clear from studies in the foundation of algebra and of geometry, acquiring an intuitive notion of the way terms may be applied and some historical feelings for the gradual adoption and adaptation of ordinary literary words to fit mathematical objectives may yet leave one with not a few vague and confused notions, many of them irrelevant to a rigorously economical logical development. It seems likely that most persons who can lead useful lives in the community have an ineradicable aversion for logical rigor, so that any course of mathematics for the average child perhaps should not dare to emphasize logical notions, particularly when they run counter to uninformed familiar practice. Whether, for able youths, logical appreciation must await physical maturity may be another question. However, many teachers are convinced that in any case it is indefensible to introduce young minds to a new topic through vague notions that are clothed in mystery and impossible to harmonize,

notions that must be unlearned later in behalf of logically simpler concepts.

Much that has been stated concerning variables will hardly survive rigorous inspection by mathematicians familiar with modern logic. To say that "a literal number is a letter that in turn is an unspecified number in an explicitly or tacitly specified set of numbers" is surely unsatisfactory. Each number is a fixed number. All or none are literal. There is, of course, no variable number slithering or slinking around in any given set of numbers. But also there is no number that is an "unspecified number." We may select some number, in perhaps an unspecified way, but the number, however, it may be chosen, is necessarily a fixed number. Furthermore, no letter can possibly be a number, constant or otherwise. A letter is at best the symbol for a number and is never the number itself. When we say "π is a number," we use, to be sure, a letter, π, called "pi." But we are not saying, "the letter π represents a number." We are talking not about any letter but about the number and using a letter as a symbol for it. Here π becomes a numeral.

We can never add a letter and a number. We may say "If n is a positive integer, then $n+1$ is a positive integer." The prefixed phrase, "for each n," is to be tacitly presupposed. But this statement, while using a letter, is not about the letter n or even about any single number. It adopts a conventional manner of speaking about the set of all positive integers; although any specified choice of an appropriate numeral, say 2 or 7, yields a corresponding particular proposition. Letters of the alphabet are not objects of actual addition in elementary algebra. To be sure, in modern algebra, we are entitled under certain strict rules to extend a number system to include stated indeterminates, but these need not concern us here.

Whenever mathematics is developed with care and with a view to logical and psychological economy, many ordinary mathematical expressions when phrased in canonical fashion become perhaps verbally cumbersome even though logically conventional. Instead of saying "Each even number is divisible by 2," we may be led to the form, "For each x, if x is an even number, then for at least one y, y is an integer, and $x=2y$." If then, in particular, it is known by some means that 6 is an even number, it also follows that "For at least one y, y is an integer and $6=2y$." In such a statement as: The positive even numbers are those in the sequence, 2, 4, $\cdots$, $2n$, $\cdots$," it is hard to define the meaning of the "$\cdots$" in terms of primitive notions.

Even on casual inspection it seems clear that while a given branch of mathematics may not be about propositions, yet the theory of such a branch is impossible to formulate save through propositions. These will be largely universal statements, referring to each member (perhaps itself a set) in some given set, but there may be numerous particular propositions such as the "number facts" of elementary arithmetic, for example "$2\times2=4$," "$3>1$," "1009 is a prime." Even this last can easily be put into the form of a universal proposition: "For each m and n, if m and n are integers and if m and n are each greater than one, then $m\times n\neq1009$."

Hardly less fundamental in mathematics than propositions are designated sets, exemplified by the set of natural numbers, the set of ten digits, the set of perfect squares, the set of all real numbers, and so forth. It would seem impossible to derive the idea of proposition from that of set, and a skeptic cannot be readily convinced that the idea of set is derivable from that of proposition. There is also the notion of symbol, and that of assertion, but these I shall not discuss.

But where does the concept of variable enter the foundations of mathematics? It is common to distinguish two uses of variable, as free variable, and as bound (or dummy, or umbral) variable. A free variable may easily be required to remain completely unrestricted, save possibly for

specification of a universe of discourse, to avoid certain paradoxes incident to ambiguity as to type. In a postulational system postulates concerning the primitive terms do indeed restrict these basic variables in certain ways. But any postulational system by a slight change in diction becomes a theory of a set of systems, and the primitive terms become then reduced to bound variables. A bound variable serves merely as a convenient but theoretically avoidable notational device for discussing a fixed set. The choice of a particular bound variable remains immaterial so long as confusion with other bound variables is avoided. A bound variable is recognized as being such by virtue of its mention in a quantifier. The quantifier is a phrase usually in one of the three basic forms or in combinations thereof; namely, (1) "For each x, $\cdots$," (the universal quantifier), (2) "For at least one x, $\cdots$" (the existential quantifier), or (3) "The set of all x, such that $\cdots$" (the designatory quantifier).

It is only when certain propositions (true or false) or certain designations are analyzed into incomplete components, that bound variables with their accompanying quantifiers are brought into evidence which, on suppressing the quantifers, appear as free variables. For illustration, consider the designatory function, x^2-3x+2. This is not a number, nor a proposition, nor a set of numbers. As written, its expression contains x as free variable. If a numeral, say, 4, is written in place of the symbol x, we obtain the expression "$(4)^2-3(4)+2$," which designates the number 6. In fact, $(4)^2-3(4)+2=6$. From this designatory function we may construct propositional functions in many ways, obtaining, for example, "$x^2-3x+2=0$," "$x^2-3x+2>0$," "$x^2-3x+2=(x-1)(x-2)$." These are not propositions, true or false statements about numbers. Even if we form the more elaborate propositional functions—(1) "If x is a real number, then $x^2-3x+2=0$," (2) "If x is a real number, then $x^2-3x+2=(x-1)(x-2)$"—unless some further quantifier is understood, the variable x is still free, and no proposition has been obtained. Each of these propositional functions may be modified to yield a designation, using in each case the appropriate designatory quantifier, thus: "The set of all x such that x is a real number and $x^2-3x+2=0$," and similarly in the other cases. But in such a completed designation, x is reduced to the status of a dummy. We may then throughout replace x by y with no change in meaning; thus the given set is also describable as "The set of all y, such that y is a real number and $y^2-3y+2=0$." Indeed this same set may be described as a certain two-membered set; and furthermore it may be completely identified by exhibition of its two members, namely, 1 and 2. In much the same way we may construct from each of these propositional functions a proposition, for example: "For each x, if x is a real number, then $x^2-3x+2=0$" (palpably false); "For each x, if x is a real number, then $x^2-3x+2=(x-1)(x-2)$" (true); and "For at least one x, x is a real number and $x^2-3x+2=0$" (true). Here again we may replace x by any other desired colorless letter. The propositions cited above are not about x, even though x is mentioned. The variable is introduced merely to facilitate a statement concerning all real numbers in connection with a certain quadratic relation. To assert that multiplication among positive integers is commutative is to assert something about the set of positive integers and the product function. No variable concepts are involved (if once the number system is accepted as fixed). But a natural manner of asserting this proposition is to assert "For each m and n, if m and n are positive integers, then $m\times n=n\times m$." In many cases of a bound variable, nothing sensible results if a known constant in the domain of the variable is substituted for the bound variable. For example, we have $(a+b)^3=\sum_{r=0}^{3} C(3, r)a^{3-r}b^r$, where $C(3, r)$ is in each case a binomial coefficient. Here r is a

bound variable. We could write i in place of r. But we should not blindly write 2 in place of r, since $\sum_{2=0}^{3}$ has no meaning.

Confusion is avoided if we recognize that the variable or unknown, x in the quadratic equation $x^2-3x+2=0$, is of the same sort as in the identity $x^2-3x+2=(x-1)(x-2)$, or in the "impossible" equation $x+1=x$. In each case, we are first specifying a certain set, namely the set of all x such that x is a real number and the given relation (in x) is valid. We then seek to identify, even to list the members of the set. Properly speaking, nothing varies in any one case. In the first case the set comprises only 1 and 2; in the second case the set consists of all real numbers; in the third case the set is void.

A given straight line in plane analytic geometry may have the equation $2x+3y-5+0$. We speak of x and y as being variables. Otherwise stated, we are identifying a certain fixed subset of the ordered pairs of real numbers, as follows. It is the set of all (x, y) such that x and y are real numbers and $2x+3y-5=0$. The set is fixed, variables x and y are bound, and serve merely as logical machinery. We may view the ordered pair (x, y) as a single (compound) variable, but still bound.

Even in the more sophisticated case of a whole postulational science, such as Euclidean geometry for which "point" and "line" remain throughout as variables, we are really asserting that each theorem in the geometry is a valid proposition for every "point" and "line" in every system of points and lines, such that the postulates are valid for "point" and "line."

It should be hardly necessary to remind the reader that a definite description while seemingly referring, as does a proper name, to an object of conceptual if not material existence, always involves (as pointed out by Bertrand Russell) two assertions, either or both of which may be contradicted by the facts. To speak of "the x such that $x+1=x$" is to start by asserting that there is one and only one x such that $x+1=x$, and then to propose that it is this uniquely existing x that is being referred to. Happily, in mathematics we do not need to bother about the varieties of tenuous subsistence enjoyed by objects no longer extant, and even perhaps never existing in the physical world of space and time, but known to us through writings. Questions can well arise about the logical existence (today) of Thomas Jefferson, Homer, Helen of Troy, Zeus, the Minotaur, Little Red Ridinghood, or David Copperfield. Mathematical objects are conceptual, impersonal, and presumably essentially as timeless as human intelligence. For most mathematicians, consistency or at most finite constructibility suffice to assure mathematical existence. But we may overdo the work of assigning conceptual models to expressions.

There is a natural temptation to read existential meanings into serviceable symbols initially introduced as incomplete parts of a larger whole. This attempt sometimes leads to fruitful images, as in the case of potential surfaces and tubes of force, of exponents, of differentials, of operators, and so forth. Often only confusion results. We should not try to separate the "$\partial^2 y$" from the "∂x^2" in "$\partial^2 y/\partial x^2$," nor the "$dx$" from the "$\int_0^1$" in "$\int_0^1 f(x)dx$," nor the "$\lim x\to 0$" from the "$f(x)=0$," in "$\lim x\to^{\circ} f(x)=0$." In "$x^2$" we may indeed speak of the exponent "2," and recently it has become customary to speak of "sin" in "sin x" as having independent meaning. But it seems safe to say that as of now a variable is not "an unspecified member of a specified set." Instead, *in any completely explicit system of mathematics a variable is a quantified (dummy) symbol in a proposition or designation.*

This note is addressed to teachers in the hope that it may prove suggestive for their thinking and teaching. However it is pitched at a mature logical level and does not pretend to give advice as to just how children should be taught, or what correct ideas are appropriate during given juvenile years. I apologize for difficulties in style, and sincerely regret any hint of

dogmatism for which the demands of brevity alone may atone.

Any teacher who has struggled through this note to its close will find pleasure and profit by reading and studying the delightfully clear text of Alfred Tarski, entitled *Introduction to Logic*, to which I am deeply indebted.[1] Many other modern reference texts on logic and the foundations of mathematics are readily available in any good library.

[1] Alfred Tarski, *Introduction to Logic* (New York: Oxford University Press, 1950).

Have you read?

NEWMAN, JAMES R., "James Clerk Maxwell," *Scientific American*, June 1955, pp. 58–71.

No student of mathematics will want to bypass the outstanding work of the physicist, James Clerk Maxwell. His use of geometry, differential equations, and general curves have provided interesting and profitable study for mathematicians and physicists since the middle of the nineteenth century. This story of his life gives the student an excellent example of how the theoretical can be perfectly blended with the practical and produce outstanding results. Maxwell won the Edinburgh Academy mathematics medal for writing a paper on constructing perfect oval curves. You will be interested in the experiments on work and sleep. You will appreciate his greatness as you read about his work with molecules and his study of astronomy which, in turn, led him to formulate Maxwell's Law of Gases.

His crowning achievement started from two papers, "On Faraday's Lines of Force" and "On Physical Lines of Force." These led to more work in electricity and resulted in Maxwell's equations which were first published in "Treatise on Electricity and Magnetism." This brief summary of his life and contributions will certainly be a challenge to your good students. Have them read it.

WILSON, JACK D. "Trends in High School Mathematics," *School Science and Mathematics*, June 1955, pp. 462–468.

It is important to identify trends even though unforeseen circumstances may well alter a trend's direction. Jack Wilson has recognized this, but at the same time he is willing to identify some of the trends his research seems to show exist.

It appears there is an increasing trend toward methods of proof. That is, what is mathematics all about? What must be defined and what left undefined? The nature of mathematical thought is coming to the foreground. The mathematical way of thinking is being applied to many non-mathematical situations. The question is, where can such thinking best be taught—in algebra, geometry, arithmetic, or integrated mathematics?

How does the trend toward creative teaching fit into this program? Will exploration and discovery by students in a mathematics laboratory allow for a higher level of attainment? Are concrete materials beneficial or detrimental?

A third trend shows algebra becoming more useful. The concept of function and dependence is becoming more important.

Other trends deal with non-segregated mathematics, motivation and differentiated instruction. However, you will want to read this article, not only to note the trends set forth but also to evaluate them as guideposts for your own future.

———"The Sturdy Age of Homespun. Part III: America's Arts and Skills." *Life*, July 18, 1955, pp. 54–68.

That mathematics is all about us historically and currently is well illustrated in this pictorial article on the age of homespun. It discusses interesting items pertaining to mathematics, such as: why patterns on homespuns were nearly always of geometric design; why cones and frustums were so much used as pitchers, jugs, and mugs. Of special interest will be the elliptical stagecoach and its relation to speed and comfort.

The section dealing with wooden clocks and their movements provides excellent examples of ratio as well as showing different numerals. It will probably surprise your students when they discover that the age of automation started in the 1790's. Although not an article on mathematics, it certainly shows the impact of mathematics on our lives.—PHILIP PEAK, *Indiana University, Bloomington, Indiana*

A new mathematical association contest

DANIEL B. LLOYD, *District of Columbia Teachers College, Washington, D.C.*

The Committee on Contests, Scholarships, and Talent Search, a Committee of The National Council of Teachers of Mathematics, reports on a successful project designed to interest high school students of mathematics.

THE MATHEMATICAL ASSOCIATION of America, Maryland–D. C.–Virginia Section, sponsored a contest in mathematics for high school students during the spring semester of 1954 and again this last year, 1955. Forty-six schools participated the first year and eighty-four the second year. A third contest for 1956 is now being planned.

All accredited high schools in the two states and the District were issued invitations to participate; this was more than 200 schools. The first year the invitations were sent out rather late and the returns were not very good; however, the 1955 acceptances were nearly double (84 schools).

The introductory statement sent to the schools set forth the following purpose of the contest: " . . . to sustain interest and encourage ability in mathematics among capable students, and also to bring the work of outstanding students to the attention of professional mathematicians and colleges throughout the area." The prospectus also included an outline of arrangements for ordering, administering, scoring, and submitting the test papers, and a brief statement concerning prizes and possible scholarship offers from colleges. A return post card was included on which a "Yes" or "No" statement of interest was elicited.

The schools replying favorably were then sent (1) a registration form, and (2) a set of contest rules. The rules were set forth as follows:

RULES FOR MATHEMATICS CONTEST

1. The contest will be an assembled written examination in high school mathematics, limited to material from elementary algebra, plane geometry, and intermediate algebra.
2. The examination will be an 80-minute examination given during the first two morning periods on Tuesday, May 4, 1954.
3. Scratch paper for use by the students is to be supplied by the school. As the examination is mainly multiple-choice, the student submits nothing but the examination booklet, but he will need paper to work on.
4. The student may be allowed to bring and use ruler and compass if he wishes, but no slide rules or other computational aids.
5. Contestants shall not consult with each other or use texts or references of any kind, and shall be seated so that they cannot see each other's papers.
6. Proctors shall not answer any questions relating to meaning or content of examination questions before or during the examination.
7. The examination shall not be given for fewer than three students.
8. The examination shall be proctored and papers scored in each school by

members of the mathematics department. Answer sheets will be furnished with the set of tests sent to each school.

9. The student submitting the highest ranking paper in each school will receive a certificate of merit from the Mathematical Association, and should be announced by the school.
10. In case of a tie for first place on the examination in any school, the department of mathematics of that school shall select the winner by consideration of the special problems on the test where work is to be shown, or if further necessary, by other discretionary means.
11. The three highest ranking papers among the boys and the three highest ranking papers among the girls will be submitted to our Committee. From these papers the highest ones from each of the three areas will be selected. A list of prizes will be presented to state and over-all winners. The names of winners will also be submitted to local colleges for possible award of scholarships.
12. Not later than March 31, a registration fee of $3 will be paid by each school entering the contest. Tests ordered in excess of thirty will be charged for at 5 cents each. A check or money order should be sent with the attached registration form to . . .
13. The Contest Committee is a board selected by the Chairman of the Maryland–D. C.–Virginia Section of the Mathematical Association of America. The actual examination has been composed by a professor at Wilson Teachers College, a mathematician at the U. S. Bureau of Standards, and a professor at the University of Maryland. At the request of the Contest Chairman, these three members are handling entirely the composing, printing, and dispatching of the papers so that the examination will be secret to all except them.

The last paragraph assured the competing schools concerning secrecy precautions, and gave assurance as to the professional conduct of the contest and the caliber of the committee members connected with it.

The problem of test construction entrusted to the three persons mentioned above was solved by our sitting down together and designing the test personally. Other possibilities had been considered, including the aid of educational testing services, etc., but it was found to be more economical and expeditious to construct it ourselves. The 1954 and 1955 tests were both constructed this way. Sample questions are shown at the end of this article. They are similar in type to the New York City Metropolitan Test, used there for a number of years. However, the total number of questions is less—40 instead of 50. One distinctive feature of our tests is the inclusion of three or four questions requiring all steps in the solution to be shown. This had several purposes: it revealed the quality and efficiency of the student's logical reasoning and his ability to present it methodically. It also served to break tied scores in certain cases.

The tests, being of objective type, were scored rather satisfactorily by the separate schools—the only exception being in the case of the problems where the steps had to be shown; some of these had to be revised by our committee. It may be observed that the tests measure aptitude as well as achievement, which can be justified in terms of the avowed purposes of the contest—to unearth and encourage potential ability and interest in mathematics.

The date set for the examinations to be given by the schools was May 4, 1954, and April 6, 1955. The May date was so late in the semester that it was difficult to get the results reported back to the schools before the end of the school year. The April date worked better, except that it happened to be the Tuesday of Holy Week, preceding Easter, during which

time a few of the parochial schools were closed. We will try to please everybody next time.

One school was taking most of its key students on a concert tour on the date set, and petitioned for a postponement. This was disallowed as the committee insisted on a uniform date for all. One school which was closed that day was permitted to give the examination in a public library suitable for the purpose.

A number of schools were disappointed with the showing made by their pupils, and two declined to submit their results for that reason. Indeed, a score of 30 or 40 did seem poor out of a possible 100. To minimize this embarrassment, the maximum score was set at 160 in 1955, instead of 100, so that resulting scores as low as 70, or even 60, looked more respectable and eliminated the chagrin of all concerned. The 1954 high score was 81 out of 100, and the 1955 high was 117 out of 160. There were not many high scores—only 20 students scored above 50 in 1954, and 28 students scored above 90 in 1955. Urban and large suburban schools seemed to score higher than the small schools. Boys predominated heavily over girls among the top scores. Fortunately, from the standpoint of dispersion of interest, the winning schools in 1954 did not repeat in 1955, except in the case of one school. One school, really on the level of a junior college, may have to be scored differently if it tends to excel unduly. This eventuality has not yet crystallized.

Judged from the number of tests ordered by each school, the schools seem to vary greatly in their use of the tests. In some schools there is apparently a thorough elimination and selection on the local level, only the very best being allowed to compete. In others, it is widely administered, and possibly used for pupil evaluation or other teaching purposes, as well as for the contest. In any event, the incentive to achievement is present and the basic purpose of the contest is realized.

The personnel of the committee conducting the contest, members of the M. A. A., happened to be distributed among the three cities, Norfolk, Washington, and Baltimore. This gave wide representation but also posed a problem in organization of duties. The chairman in Norfolk did the planning and much of the correspondence with the schools, but he also delegated considerable responsibility to other committee members. For instance, the Washington group constructed and distributed the tests. The scored tests were mailed back to one member there, who consolidated the results, checked the scoring of the ones nearest the top in each state, and submitted the summary of winners to the awards subcommittee. This committee, in turn, made awards to winners and runners-up in each state, and sent names of highest winners to colleges which were offering scholarships. One committee member mailed out the certificates of merit to the highest student in each competing school. This was an engraved certificate, bearing the seal of the Mathematical Association of America.

The test papers were multilithed on $8\frac{1}{2}'' \times 11''$ sheets, stapled together. Answer sheets were furnished each school. The schools scored the papers and submitted the names of the three best boys and the three best girls—since there was a possibility of a scholarship being offered to only a boy, or only a girl, by certain colleges. Only four colleges so far are offering scholarships.

Besides the Certificate of Merit presented to the high scorer in each school, there were over-all and state awards. The M. A. A. gave a $25 savings bond to each of the three state winners, with another $50 bond to the over-all contest winner, which this year was donated by the Bendix Aviation Corporation. For the runners-up, a number of attractive and useful articles were donated by other firms and business establishments. One college gave a copy of Newton's *Principia* (Mott's translation), and an affiliated group of The National Council of Teachers of

Mathematics made a monetary contribution.

After the contest a summary of results was published and sent to the schools, giving the list of winners and their scores. This allayed much mystery and wonder, and served as an incentive for the following year. A more complete story of the contest was issued as a press release to the newspapers in the larger cities, which, however, generally required some prodding to assure publication. Large urban dailies seem to have so many more "interesting" things to occupy their columns.

The committee members conducting the contest were: Mr. William H. Norris, Norfolk Public Schools, chairman; Dr. Evelyn Boyd, mathematician, U. S. Bureau of Standards; Dr. Stanley B. Jackson, University of Maryland; Rev. C. F. Koehler, Loyola College, Baltimore; Dr. Daniel B. Lloyd, District of Columbia Teachers College, Washington; Miss Carol V. McCamman, D. C. Public Schools; Mr. Paul L. Chessin, of Westinghouse Corporation (Mr. Chessin was particularly effective in securing donations and prizes from Baltimore manufacturers). Dr. Francis E. Johnston, of George Washington University, section president of the association, and Dr. Richard P. Bailey, of the U. S. Naval Academy, section secretary, coordinated the awarding of the prizes.

Following are sample test questions taken from the 1955 examination:[1]

1. If x represents a number for which $x<0$, then
   (a) $x>0$ (b) $-x<0$ (c) $-x=0$ (d) $x=0$ (e) $-x>0$
2. If the side of a square is 11 inches, correct to the nearest inch, how many significant digits can one be absolutely certain of in the computed area?
   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4
3. How many inches long is a fish if the head is 8 inches long, the body twice as long as the tail, and the tail is the length of the head plus one-half the length of the body?
   (a) 20 (b) 24 (c) 30 (d) 36 (e) No solution
4. $\frac{\log 9}{\log 3}$ equals
   (a) 2 (b) log 2 (c) 3 (d) log 3
   (e) log 9 − log 3
5. To contradict the statement that "For every boy there is a girl" is equivalent to sta ting
   (a) For some boy there is no girl.
   (b) For every girl there is a boy.
   (c) For no boy is there a girl.
   (d) For some girl there is no boy.
   (e) None of these.
6. Given the statement "If John can vote, then John is over 21," which of the following statements is then necessarily true?
   (a) If John cannot vote, he is not over 21.
   (b) If John is over 21, he can vote.
   (c) Either John can vote or John is not over 21.
   (d) Either John can vote or he is over 21.
   (e) If John is not over 21, he cannot vote.
7. A right triangle has sides of 18, 24, and 30 inches, respectively. The diameter of the inscribed circle is between
   (a) 0 and 8 (b) 8 and 9 (c) 9 and 10
   (d) 10 and 11 (e) 11 and 20

[1] Copies of the 1955 examination may be obtained by sending 25 cents to Daniel B. Lloyd, D. C. Teachers College, Georgia Ave. and Euclid St., N.W., Washington, D. C

United States complacency hit

Continued from page 463

One of the youngest of the leading Soviet mathematicians is I. G. Petrovsky, who likewise has been named to the Academy of Sciences. Mr. Petrovsky is also one of the most highly decorated members of the Soviet school. For his studies in aerohydrodynamics he received a Stalin Prize.

L. S. Pontryagin, who has been sightless since boyhood, has become, nonetheless, one of the world's foremost topologists. Well known as an abstractionist, Mr. Pontryagin also enjoys a high reputation for his work in differential equations.

ARDENT MARXIST

The only non-Soviet mathematician in this list is Philip Frankl, a middle-aged Communist from Vienna. An ardent believer in Marxism, Mr. Frankl deliberately moved to Moscow some time ago. His strong point is hydrodynamics.

Even this brief list makes it clear that Americans should neither underestimate the native ability of Soviet mathematicians nor underrate the capabilities of hundreds of top-notch scholars who are now taking advantage of, and benefiting by, the fact that their chosen profession is today in a much stronger position than it was during the days of the Czars.—*Reprinted by permission of the author, Frank Rounds, Jr., and* The Christian Science Monitor.

• DEVICES FOR THE MATHEMATICS CLASSROOM

Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

Finding distances with the telemeter

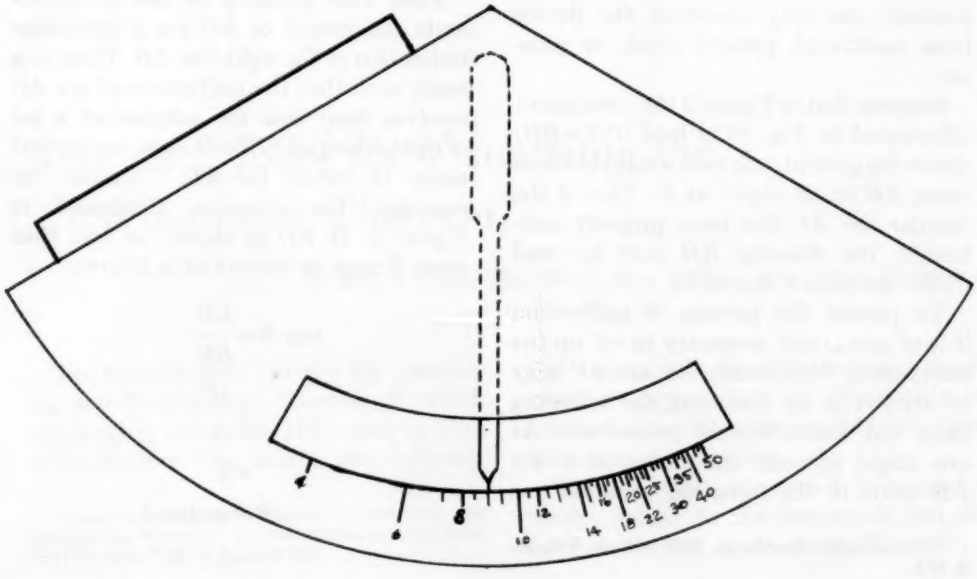
by Herbert J. Schiff, Gage Park High School, Chicago, Illinois

The telemeter is an instrument which may be used to estimate the horizontal distance between two points. Its construction depends on the application of concepts from trigonometry, but the instrument itself may be used in almost any high school mathematics class. Stone and Mallory suggest the name "telemeter" for the particular type of distance-meter which they describe in their book *New Plane Geometry*.[1] However, the term "telemeter" may be ascribed to any one of a variety of instruments designed to measure distances between objects. *Encyclopedia Americana* includes the following entry on the topic:

> Telemeter, any one of the various instruments designed to measure the distance of objects more or less remote. It is used by surveyors and engineers, where rapid work is necessary and a fair amount of accuracy required, on the battlefield to determine ranges, etc., where it is usually called a range finder. One form is a recording apparatus electrically connected with a distant meteorological or other instrument. The surveyor's telemeter is better known as a stadia, and consists of a mounted telescope with horizontal cross-wires, which, when in

[1] John C. Stone and Virgil S. Mallory, *New Plane Geometry* (Chicago: Benjamin H. Sanborn and Company, 1940), p. 334.

Figure 1



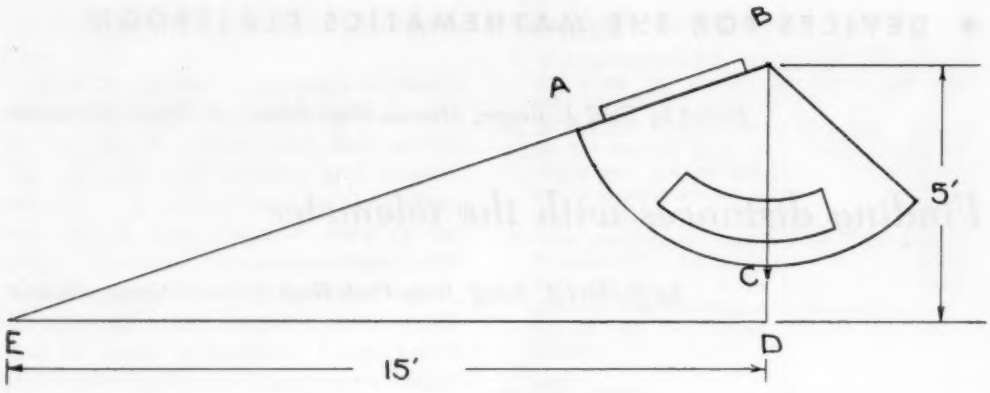


Figure 2

use, intercept the divisions upon a graduated rod held by an assistant at the distant point. The reading multiplied by the "factor" of the instrument gives the distance.[2]

An additional fact which deserves mention is that military range finders usually involve the use of lenses and double images similar to the range finders built into quality cameras.

The device described in this article is not a complicated one to produce. Essentially it is patterned after the instrument proposed by Stone and Mallory; however, there are some modifications and additional features (Fig. 1). Depending on the degree of excellence desired in the finished product, one may construct the device from cardboard, pressed wood, or plastic.

Suppose that in Figure 2 the instrument (illustrated in Fig. 1) is held 5′ ($=BD$) above the ground, and that a sight is taken along AB on an object at E. Then if the circular arc AC has been properly calibrated, the distance ED may be read under the plumb line at C.

To pursue the process of calibration it is of course not necessary to set up the instrument. Scale marks on arc AC may be arrived at by observing the following facts and computational procedures. As one might suspect, the choice of 5′ for BD made in the foregoing example was arbitrary. The height BD may be selected to suit one's taste; however, only one instrument can be developed for a given choice of BD. This is true because the computations by which the scale marks on arc AC are determined depend on the fact that for all triangles EBD, side BD is the same. It goes without saying that this height must be observed when one uses the instrument to find unknown distances. To avoid any variation in BD when sighting on objects, it is advisable to mount the instrument on a tripod so that BD will have the constant length selected (Fig. 3).

Each scale marking on arc AC represents the length of ED for a particular inclination of the sight line AB. Thus, it is easily seen that the calibration of arc AC resolves itself into the solution of a set of right triangles EBD for some convenient range of values for ED. Consider, for example, the situation illustrated in Figure 2. If ED is chosen as 15′, then angle B may be computed as follows:

$$\tan B = \frac{ED}{BD}$$

$$= \frac{15'}{5'}$$

$$= 3$$

$$B = \arctan 3$$

$$= 71° 33'.$$

[2] *Encyclopedia Americana*, 1954 edition, Vol. 26, p. 372.

Figure 3

Now with the aid of a protractor, mark off the terminal side of angle B on arc AC and write *15′* on the scale. The procedure described above should be repeated until one has located as many scale markings as are desired. The range of the instrument has a maximum upper limit of about 75′ because it is impossible to achieve much accuracy when angle B is near 90°. Students, in general, are impressed with this application of right-triangle solution and derive much satisfaction from calibrating an instrument.

The physical construction of the device may be readily effected by using poster board for the body, a small paper tube for sighting and a lead-weighted cardboard arm for the plumb line. The arm may show through a slit cut in the body or may hang directly over the front of the instrument. If a more permanent device is desired, pressed wood or plastic should be employed.

Other classes besides those in trigonometry may profit from the study and construction of this device. The usual work in numerical trigonometry pursued in plane geometry should provide the students in these classes with enough background to enable them to compute the angles needed for constructing the instrument. Moreover, classes in general mathematics should be able to build an instrument of this type if the plans and sizes of appropriate angles are provided. Students will be thrilled by the "magic" ability of measuring distances simply by sighting along an edge.

Note on "Model for Visualizing the Pythagorean Theorem"

by Edwin Eagle, San Diego State College, San Diego, California

The contribution, "Model for Visualizing the Pythagorean Theorem,"[3] which appeared in the April, 1955 issue of THE MATHEMATICS TEACHER is very interesting. I agree with the comment that pupils gain in appreciation of the beauty of mathematics if they are taught to visualize geometric relationships.

For a number of years I have used a model similar to the one which you described in the April issue. However, I

[3] Emil J. Berger, "Model for Visualizing the Pythagorean Theorem," THE MATHEMATICS TEACHER, XLVIII (April 1955), pp. 246–247.

believe that the model which I have been using may possibly have some advantages over the one proposed by you. You suggest painting the three squares gray. Using the figure as labeled in your article, I find it better to have square AS and the equivalent rectangle AH painted in one color, and square BY and its equivalent rectangle BH painted in another color (see Fig. 4). This little revision makes it easier to talk about the figures involved and should help students discover essential relations more easily. Also, instead of using *four* semi-transparent plastic oblique triangles, as you suggest, I use *two*. Simply pivot triangle BAT at A and triangle ABX at B. If this is done, these two triangles can be rotated to assume respectively the positions of triangles RAC and MBC.

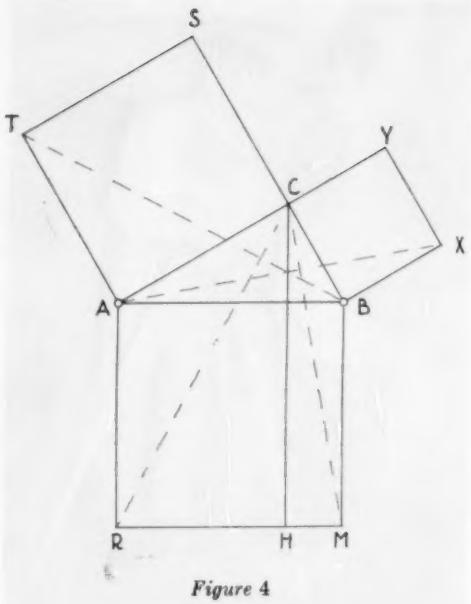


Figure 4

Have you read?

McCracken, Daniel D. "The Monte Carlo Method," *Scientific American*, May 1955, pp. 90–96.

This article is not about that famous casino. The probability here deals with neutrons, needles, and furniture factories with roule'te wheels for computational purposes. The Los Alamos Scientific Laboratory gave birth to this method as a result of attempting to solve a rather knotty problem about neutrons. The method was the work of two mathematicians, John Von Newmann and Stanislas Ulam, who gave it the name "Monte Carlo."

If you select a short needle and draw on a paper parallel lines twice the length of the needle apart and then toss the needle on this sheet, the ratio of the number of times the needle touches a line to the number of tosses will be 1 to Pi. This is not new, but the real interest comes from the use of the roulette wheel and random members to replace the random trials. In this way the effect of many trials can be found and calculated. The procedure gives approximate results but the degree of precision can be determined by the number of trials.

This method opens the way for many possibilities on research where the variables are not adaptable to mathematical formulation. The author uses a furniture factory as a good example. You will be interested in considering the future of the Monte Carlo method.

Moore, Alexander, "It's a Challenge for Every Teacher," *Indiana Teacher*, April 1955, pp. 394–395.

Who is responsible for solving the juvenile delinquency problem? How can every teacher help? What should be the approach? This article gives a committee's view, and, whether you agree or disagree, it makes us more intelligently aware of what is happening. Is there too much pressure for action without analysis? Is it sentimentality or excuses?

The committee presents some of the things they think teachers can and should do; for example, work as individuals through local agencies; examine their teaching and school activities.

Above all, educators must perpetuate optimism, belief and faith. I think this short simple article is good reading for us all.—Philip Peak, *Indiana University, Bloomington, Indiana*

• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Mathematics on stamps

by H. D. Larsen, Albion College, Albion, Michigan

An interesting classroom project overlooked by most mathematics teachers is the collecting of postage stamps that have reference to mathematics. Such stamps provide excellent material for bulletin boards as well as topics for interesting club programs.

Nearly every town has one or more stamp collectors who stand ready to advise a beginner on the proper way of mounting stamps, etc. One needs a pair of stamp tongs and a packet of stamp hinges which are available from stationery stores at a nominal cost. The stamps themselves may be purchased from stamp dealers, usually for a few cents each. Every large city has at least one stamp dealer ready to serve collectors.

The accompanying photograph displays some of the many stamps having particular reference to mathematics. The first stamp displayed was issued early in 1955 to commemorate the adoption in France of the metric system of weights and measures. The allegorical figure is pictured measuring the quadrant of the earth, the original basis for the meter. The second stamp displayed is a recent German issue, commemorating the centenary of the death of Carl Friedrick Gauss, one of the foremost mathematicians of all times.

A list of stamps carrying portraits of mathematicians[1] has been published elsewhere. A revised list is given below for the convenience of readers of THE MATHEMATICS TEACHER, who may wish to start a collection of stamps of mathematical interest. Catalog numbers and descriptions cited are those in *Scott's Standard Postage Stamp Catalogue, 1955.* I will be grateful if I am notified of any stamps that have been overlooked.

Abel, Niels Henrik (1802–1829)
Norway, 1929:
145. 10 ö green
146. 15 ö red brown
147. 20 ö rose red
148. 30 ö ultramarine

Ampere, Andre Marie (1775–1836)
France, 1936:
306. 75 c brown
France, 1949:
626. 15 fr sepia

Armero, Julio Garavito (1865–1920)
Colombia, 1949:
573. 4 c green

Avicenna (979–1037)
Afghanistan, 1951:
390. 35 p deep claret
391. 1.25 f blue
Lebanon, 1948:
223. 30 pi orange brown and buff
224. 40 pi Prussian green and buff
Germany, 1952:
10N106. 35 pf blue
Poland, 1952:
558. 75 g red brown

Bolyai, Farkas (1775–1856)
Hungary, 1932:
479. 70 f cerise

Boscovich, Roger Joseph (1711–1787)
Croatia, 1943:
59. 3.50 k copper red
60. 12.50 k dark violet brown

Brahe, Tycho (1546—1601)
Denmark, 1946:
300. 20 ö dark red

Chaplygin, Sergei A. (1869–1944)
Russia, 1944:

[1] Carl B. Boyer, "Mathematics and Philately," *Scripta Mathematica*, Vol. 15 (June 1949), pp. 105–114. H. D. Larsen, "Mathematics and Philately," *American Mathematical Monthly*, Vol. 60 (February 1953), pp. 141–143.

Left to right, top row: Metric commemorative, Gauss, Copernicus, Pascal. Second row: Descartes, Galileo, Da Vinci. Third row: Abel, Euler, Hamilton, Leibniz. Bottom row: Tycho Brahe, Chebychev, Poincaré, Simon Stevin.

945. 30 k gray
946. 1 r light brown

Chebychev, Pafnutiy Lvovich (1821–1894)
Russia, 1946:

1050. 30 k brown
1051. 60 k gray-brown

Condamine, Charles-Marie de la (1701–1774)
Ecuador, 1936:

347. 2 c blue
348. 5 c dark green
349. 10 c deep orange
350. 20 c violet
351. 50 c dark red

Copernicus, Nicholas (1473–1543)
Poland, 1923:
192. 1000 m indigo
193. 5000 m rose
Poland, 1942–43:
NB23. 1 z+1 z dull myrtle green
NB27. 1 z+1 z rose lake
Poland, 1951:
515. 1.15 z claret
Poland, 1953:
578. 20 g brown
579. 80 g deep blue

Descartes, René (1596–1650)
France, 1937:
330. 90 c copper red
331. 90 c copper red

Dürer, Albrecht (1471–1528)
Germany, 1925:
362. 80 pf chocolate

Eötvös, Lorand (1848–1919)
Hungary, 1932:
471. 6 f yellow green
Hungary, 1948:
840. 60 f deep red

Euler, Leonard (1707–1783)
Germany, 1950:
10N58. 1 pf gray

Galilei, Galileo (1564–1642)
Italy, 1933:
D16. 35 c rose red
Italy, 1942:
419. 10 c dark orange and lake
420. 25 c gray green and green
421. 50 c brown violet and violet
422. 1.25 l Prussian blue and ultramarine
Italy, 1945:
D18. 1.40 l dull blue

Gerbert, i.e., *Pope Sylvester II* (c. 950–1003)
Hungary, 1938:
511. 1 f deep violet
516. 10 f red orange

Gauss, Carl Friedrich (1777–1855)
Germany, 1955:
729. 10 pf dull myrtle green

Gazeta, Matematica
Romania, 1945:
596. 2 l sepia
597. 80 l blue-black

Hamilton, Sir William Rowan (1805–1865)
Irish Free State, 1943:
126. ½ d deep green
127. 2½ d dark red brown

Huygens, Christiaan (1629–1695)
Netherlands, 1928:
B36. 12½ c+3½ c ultramarine

Jacobsen, Jacob Christian (1811–1887)
Denmark, 1947:
304. 20 ö dark red

Kovalevskaya, Sonya (1850?–1891)
Russia, 1950:
1570. 40 k purple on salmon

Leibniz, Gottfried Wilhelm (1646–1716)
Germany, 1926:
360. 40 pf deep violet

Lobachevski, N. I. (1793–1856)
Russia, 1950:
1575. 40 k brown

Lorentz, Hendrik Antoon (1853–1928)
Netherlands, 1928:
B35. 7½c+3½c vermillion

Mercator, Gerard (1512–1594)
Belgium, 1942:
B324. 1.75 fr+50 c dull blue

Ortelius, Abraham (1527–1598)
Belgium, 1942:
B325. 3.25 fr+3.25 fr lilac rose

Ostrogradski, M. V. (1801–1861)
Russia, 1951:
1604. 40 k black brown on pink

Pascal, Blaise (1623–1662)
France, 1944:
B181. 1.20 fr+2.80 fr black

Poincaré, Henri (1854–1912)
France, 1952:
B279. 18 fr+5 fr dark brown

Roemer, Ole (1644–1710)
Denmark, 1944:
293. 20 ö henna brown

Stevin, Simon (1548–1620)
Belgium, 1942:
B321. 50 c+10 c fawn

Swedenborg, Emanuel (1688–1772)
Sweden, 1938:
264. 10 ö violet
266. 10 ö violet
267. 100 ö green

Teixeira, Francisco (1851–1933)
Portugal, 1952:
751. 1 e cerise
752. 2.30 e deep blue

Vinci, Leonardo da (1452–1519)
Italy, 1932:
C29. 1 l violet
C30. 3 l brown red
C31. 5 l deep green
C33. 10 l+2.50 l black brown
C34. 100 l bright blue and greenish black
Italy, 1935:
347. 50 c purple
348. 1.25 l dark blue
Italy, 1937:
404. 50 c light violet
C103. 2 l royal blue
C105. 5 l deep green
Italy, 1951:
601. 25 l deep orange
Liechenstein, 1948:
C24. 10 rp dark green
France, 1952:
682. 30 fr deep ultramarine
Germany, 1952:
10N104. 20 pf green
Hungary, 1952:
C109. 1.60 fo dark Prussian green
Poland, 1952:
B73. 30 g+15 g ultramarine
Romania, 1952:

878. 55 b purple
Witt, Johan de (1625–1672)
Netherland, 1947:
B177. 7½c+2½c dark purple brown
Zhukovsky, Nikolai (1847–1921)
Russia, 1941:
838. 15 k deep blue
839. 30 k carmine rose
840. 50 k brown violet
Russia, 1947:
1098. 30 k sepia
1099. 60 k blue violet

A practical application of mathematics

By Robin Biggs, pupil, Meadville High School
Meadville, Pennsylvania

I, digging Math books
From out of dusty attics,
Know a girl who got her man
By means of mathematics.
Before she started planning,
The dreadful thought would dwell,
The awful possibility
Their paths were ∥.
What's so horrible you ask?
Only this my sweet,
No matter how extended
∥s can *never* meet.
So she planned her first attack,
(She should have had more sense.)
She tried her best to trap him with
A circumference.
She thought then catching him would be
Easier than π
Alas, she lost her lover
Among the radii.
She found a circle's larger
Than it sometimes seems to be,
She saw him hiding deep behind
The forty-first degree.
At this striking setback
Some would have stopped and cried,
But it only found her efforts
Doubly remultiplied.
He tried all the figures
Her advances to avoid;
First the square and △,
Then the trapezoid.
But to this she gave no heed
And would not stop the wrangle;
And no pun's intended when I say
She tried another ∠.
And I'm afraid his far-flung flights
Did not at all surprise her,
Of cunning plots and subplots
She was a cool divisor.
So if you think it's horrible—
All that time to spend—
You must admire her foresight,
And Man! the dividend!
Caught at last, his defense
Was now of no duration;
She had the terms; now she could start
Forming the equation.
During the ceremony
The groom began to fidget;
Instead of answering, "I do,"
The goof replied, "I digit."
However, I guess it could be said
(And I allow no laughter!)
Their life till now could be summed up
As "happily ever after."
So if your mind is twisted
In mental acrobatics;
Be like me. Dig out those books.
Let's *use* our mathematics!!!!

• MATHEMATICAL MISCELLANEA

Edited by Paul C. Clifford, State Teachers College, Montclair, New Jersey, and Adrian Struyk, Clifton High School, Clifton, New Jersey

Integral squares with square sum

by Francis L. Miksa, Aurora, Illinois

Francis L. Miksa, one of whose tables[1] was published in this department earlier this year, has sent us some interesting examples of a method he devised years ago for determining integral squares whose sum is a square. The procedure is based upon a general identity which may be written

$$(2a_r)^2(S-a_r^2)+(S-2a_r^2)^2=S^2,$$

where a_r denotes any one of n quantities $a_1, a_2, \cdots, a_n$ whose squares have the sum S. Particular cases are

$$(2ab)^2+(b^2-a^2)^2=(a^2+b^2)^2,$$
$$(2ba)^2+(2bc)^2+(a^2-b^2+c^2)^2=(a^2+b^2+c^2)^2.$$

The substance of Mr. Miksa's note follows.—*Editor*

I shall explain some examples in detail, applying the method first to

$$(1) \qquad w^2+x^2+y^2+z^2=t^2.$$

Set up four columns thus:

I	II	III	IV
1	2	1	49 (51− 2)
3	6	9	33 (51−18)
5	10	25	1 (51−50)
4	8	16	19 (51−32)
		51	

In the first column list any four integers in any order. The entries in column II are double those in I. The entries in III are the squares of those in I; the sum of these squares is also recorded. To complete column IV subtract twice each square in III from the sum in III. Four solutions of (1) are now readily obtained. Take any integer from II and multiply the other integers of I by it. Take from IV the integer corresponding to that taken from II. Take the sum from III. It is convenient to symbolize these selections as follows:

2(3, 5, 4), 49; 51. 6(1, 5, 4), 33; 51.
10(1, 3, 4), 1; 51. 8(1, 3, 5), 19; 51.

Expressed in the form of equation (1) the results are

$$6^2+10^2+8^2+49^2=51^2,$$
$$6^2+30^2+24^2+33^2=51^2,$$
$$10^2+30^2+40^2+1^2=51^2,$$
$$8^2+24^2+40^2+19^2=51^2.$$

The method is easily applied to a greater number of terms. Here is an example of summations of six squares:

[1] F. L. Miksa, "Table of integral solutions of $a^2+b^2+c^2=r^2$ for all odd values of r from $r=3$ to $r=207$," THE MATHEMATICS TEACHER, XLVIII (April 1955), pp. 251–255. Subsequent computations have extended this table to $r=325$. Further results (unpublished) of computations which have been undertaken by Mr. Miksa include extensive sets of isoperimetric and of equiareal primitive Pythagorean triangles; counts of such sets for all perimeters less than ½ million and for all areas less than 10 billion; binomial coefficients for all powers up to the 100th; a table of integral solutions of $x^2+y^2=N$ for all odd N less than 10^4.

I	II	III	IV
1	2	1	167
3	6	9	151
5	10	25	119
7	14	49	71
9	18	81	7
2	4	4	161
		169	

In symbolic form the six results now obtainable are

$$2(3, 5, 7, 9, 2), 167; 169.$$
$$6(1, 5, 7, 9, 2), 151; 169.$$
$$10(1, 3, 7, 9, 2), 119; 169.$$
$$14(1, 3, 5, 9, 2), 71; 169.$$
$$18(1, 3, 5, 7, 2), 7; 169.$$
$$4(1, 3, 5, 7, 9), 161; 169.$$

Thus each of the following sums is equal to 169^2:

$$6^2+10^2+14^2+18^2+4^2+167^2,$$
$$6^2+30^2+42^2+54^2+12^2+151^2,$$
$$10^2+30^2+70^2+90^2+20^2+119^2,$$
$$14^2+42^2+70^2+126^2+28^2+71^2,$$
$$18^2+54^2+90^2+126^2+36^2+7^2,$$
$$4^2+12^2+20^2+28^2+36^2+161^2,$$

A rather amazing outcome of the method under consideration appears in the following example, where $i=\sqrt{-1}$.

I	II	III	IV
$3i$	$6i$	-9	83 $(65+18)$
5	10	25	15 $(65-50)$
7	14	49	-33 $(65-98)$
		65	

The symbols of the proper combinations are

$$6i(5, 7), 83; 65.$$
$$10(3i, 7), 15; 65.$$
$$14(3i, 5), 33; 65.$$

The first result obtained is

$$-30^2-42^2+83^2=65^2$$

or

$$83^2=65^2+42^2+30^2.$$

From the remaining two combinations the rather unexpected results are

$$14^2+3^2=13^2+6^2,$$
$$70^2+33^2=65^2+42^2.$$

Finally, I shall exemplify another application of the method by determining a solution of the equation

$$x^2+y^2+z^2=3(u^2+v^2+w^2).$$

Set it up this way:

I	II	III	IV
$2\sqrt{3}$	$4\sqrt{3}$	12	38
3	6	9	44
4	8	16	30
5	10	25	12
		62	

Only the combination $4\sqrt{3}(3, 4, 5), 38; 62$ is to be used. Then

$$3(12^2+16^2+20^2)+38^2=62^2,$$

and this reduces to

$$3(3^2+4^2+5^2)=150.$$

If, now, 150 can be expressed as the sum of three squares, the problem is solved. But

$$150=1^2+7^2+10^2=2^2+5^2+11^2$$
$$=5^2+5^2+10^2.$$

Therefore,

$$1^2+7^2+10^2=3(3^2+4^2+5^2).$$

The procedure that has been described above is not one suited to systematic computation of all possible solutions of a given type of equation. Consider the equation

$$a^2+b^2+c^2+d^2=65^2, \tag{2}$$

which has a total of 127 integral solutions.[2] From a table of integral solutions of $x^2+y^2=n$ it is readily apparent that

$$2^2+3^2+4^2+6^2=65$$

is the only way to express 65 as a sum of four squares. Hence, using 2, 3, 4, 6 in column I, we find that

[2] To obtain a table of these 127 solutions, send a stamped self-addressed envelope to A. Struyk, Clifton High School, Clifton, N. J.

(12, 16, 24, 57), (12, 24, 36, 47),
(16, 24, 48, 33), (24, 36, 48, 7),

are the only four sets (a, b, c, d) of the 127 solutions of (2) that result directly from integral entries in that column. A solution easily obtained by using Pythagorean triangles is

$$15^2+20^2+36^2+48^2=65^2,$$

the expansion of

$$(3^2+4^2)(5^2+12^2)=5^2\cdot 13^2.$$

We get this as a result of the column method if we use 50, 20, 36, 48 in column I, since one of the equations obtained has 100 as a common factor of its terms. Other column-I entries yielding integral solutions of (2) are the numbers $4\sqrt{2}$, $2\sqrt{2}$, $1/\sqrt{2}$, $7/\sqrt{2}$, and the numbers $\sqrt{2}$, $1/\sqrt{2}$, $5\sqrt{2}$, $5/\sqrt{2}$. The chief merit claimed for the procedure is its ease of application in determining specific examples of various relationships involving sums of squares.

Have you read?

DENBOW, C. H., "Postulates and Mathematics," *The American Mathematical Monthly*, April 1955, pp. 233–236.

Here is an article which should be required reading for all of us who received our training some years ago and have failed to keep in close contact with the growing mathematics. Postulationism can lead to a disastrous breakdown in communication between specialist and layman. True, the abstract approach often is the more efficient and it also allows for a logical system. But isn't there a time when logic and efficiency need to be replaced with a combination of the logical and the empirical? This seems to be necessary to assure communication between instructor and student or layman. Certainly some of mathematics is an intimate mixture of mental power and physical reality. Concepts grow out of experience but idealizations do also. Whether you wish to accept this "compromise view" on the nature of mathematics or not, this article certainly makes one conscious of the existing problem.

MAYER, JOSEPH, "Mathematical Ideas," *School Science and Mathematics*, January 1955, pp. 5–13.

All teachers of mathematics recognize the value of working with ideas in mathematics. Mr. Mayer in his brief article has presented a series of mathematical ideas in a very logical manner. His thesis is that students often have an aversion to mathematics because of unpreparedness to comprehend the symbolism and its meaning.

The ideas presented are not new, but it is always well to rethink them and make more effective use of them. Some of these ideas are: the over-all generalizing character of mathematics and the idea of number with all their aspects in order, summation, positive and negative, more than and less than, and ratio.

The ideas are related to rules of operation both with specific and general numbers, so general, in fact, that they carry the name imaginary. These ideas lead to those of spatial relations, order, point, plane, continuity. From here one proceeds to stationary or moving points and on into calculus. But the past 100 years have shown great advancement in the development of these ideas. Such ideas should always be before our mathematics students so they may look into the future and carry on.

NELSON, A. GORDON, "Better Teacher-Student Relations," *Phi Delta Kappan*, May 1955, pp. 295–302.

No one disagrees with the principle that attitudes are important as a contributing factor to teaching success. Did you ever stop to consider how these attitudes come about? What factors go into a composite which results in an attitude? This article deals with some of the clues and opinions about the problem. What are some effects of the mental health of the teacher, and what constitutes good mental health? Are factors of cooperation, patience, humor, appearance, consideration for the individual, consistent behavior and the use of praise valuable in building a teacher-student relationship which is conducive to efficient learning? Can students recognize good teaching when they are in such a class? What can the public and the administration do for morale in the profession of teaching? Do the purposes of education prevent or assist in developing desirable attitudes? Are there ways of obtaining information, which helps a teacher know and become interested in his students? Of how much value is this interest to student-teacher relations? What are some of the problems which arise when communication between teacher and student breaks down? These and many other questions about teacher-student relationships are discussed in this article. You are already carrying out many of the suggestions, but there are others in which you will be interested.
PHILIP PEAK, *Indiana University, Bloomington, Indiana*

Edited by Lucien B. Kinney, School of Education, Stanford University, and Dan T. Dawson, School of Education, Stanford University, Stanford, California

Using mathematical recreations in the junior high school

by Arthur J. Hall, San Francisco State College, San Francisco, California

What are mathematical recreations? What good are they? Where can they be found? How are they most effectively used? What research has been done on their use and effectiveness? These were the questions that an in-service seminar for mathematics teachers, under the direction of the writer, undertook to answer.

There are numerous occasions in the junior high school classroom where the use of recreations is good strategy. Brandes has indicated a number of these.[1] Generally, recreations may be used as (1) a part of the lesson plan or as a recreation that would be an excellent problem and is clearly related to the day's work; (2) a presentation to an individual or small group to stimulate interest in mathematics; (3) a reward to those pupils completing their assignments; (4) a part of the daily work; (5) a focus of attention at the beginning of a class period that brings the class together; (6) a theme for bulletin board displays; (7) a filler for those sometimes-unplanned vacuums near the end of a class period; (8) a theme for special days; (9) a filler for the day or days when half the class is absent because of illness, musical programs, or special events. Mathematics clubs have also used recreations as part of their programs.

[1] Louis G. Brandes, "Using Recreational Mathematical Materials in the Classroom," THE MATHEMATICS TEACHER, XLVI (May 1953), pp. 326–9.

The outcome of strategic use of recreations not only includes a better attainment of the aims of mathematics relating to attitudes and appreciation, but also a greater achievement of the other objectives. This is no doubt due, in part, to recreations promoting a wholesome learning situation and also providing a powerful means of instruction. Important learning facilitators such as these are in evidence: (1) a better teacher-pupil relationship; (2) new worlds of mathematics quickly and sometimes dramatically opened to the pupils; (3) bringing to life dull but important aspects of traditional mathematics; (4) a useful but extremely flexible classroom activity that the teacher can fit into a variety of situations or use in providing a change of pace.

It is difficult to make a sharp distinction between recreational and non-recreational material. Generally, a mathematical recreation is some intriguing mathematical problem without utilitarian overtones, such as a number oddity; historical information concerning symbols or great mathematicians; trick problems depending upon clever manipulations of numbers, symbols, or relationships; fallacies; manipulation of numbers including magic squares and short cut methods of calculations; problems in spatial relations; algebra in numerous ways—such as "think of a number, double it, add 2, . . . etc."; and absurdities resulting from division by zero.

Finding mathematical recreations is easy. The interested reader is referred to the departments "Mathematical Recreations," "Devices for the Mathematical Classroom," and "Historically Speaking,—" in issues of the THE MATHEMATICS TEACHER. Textbooks on methods, such as those by Butler and Wren,[2] and Kinney and Purdy[3] also contain extensive bibliographies on recreations, with suggestions for their use.

Several types of recreations discovered by the seminar are very popular both among pupils and teachers in the junior high school, such as:

In a certain town 8 per cent of the inhabitants are one-legged and 50 per cent of the remainder go barefooted. How many shoes does it take to shoe all the inhabitants in town? Upon arriving at the answer (one shoe per inhabitant, or number of shoes equals population), the percentages may be altered and the answer again determined. This, with a little thought, leads to the interesting conclusion that the problem can be stated as "x per cent are one-legged" with the same general result.

With six matches, form four equilateral triangles. (Many people miss this one because they are not thinking of three dimensions or tetrahedrons.)

Write the equivalent of 78, using four sevens. (77 7/7)

Lattice multiplication and the Fool's Rule were the most interesting of the older algorisms. Another recreation that proved exceedingly popular was a story "Paul Bunyan versus the Conveyer Belt." It is too long to reproduce here but may be found in the *Ford Times* (July 1949). The story is a clever presentation of the result of cutting a Mobius strip twice. An actual demonstration with the reading of the story is fascinating to the pupils.

One algebra teacher used a series of related problems to show the power of symbolic thinking and the importance of using ideas learned in one problem to do another. Several days intervened between the problems. The lead problem is an old favorite.

Assume the earth is a sphere with a circumference of 25,000 miles and a wire stretched around the earth at the equator. The wire is cut in one place and 10 feet of wire added. The wire is then stretched around the earth so that it is equidistant from the earth at all points. How far is the wire from the earth? The pupils characteristically start with an arithmetical approach and essentially do the following in piecemeal fashion:

$$X=\frac{25{,}000\times 5280+10}{2\times 3.14}-\frac{25{,}000\times 5280}{2\times 3.14}.$$

This teacher stopped the class before they completed the calculations but after all had become aware of the time required to do the problem in this fashion. She then drew from the class the following solutions:

r = radius of earth and original wire
R = radius of wire after 10 feet added
$R-r$ = distance of wire from earth
C = circumference of earth
C' = circumference of wire after 10 feet added

$$C'=2\pi R \text{ or } R=\frac{C'}{2\pi}$$

$$C=2\pi r \text{ or } r=\frac{C}{2\pi}$$

$$R-r=\frac{C'}{2\pi}-\frac{C}{2\pi} \text{ or } \frac{C'-C}{2\pi}$$

But $C'-C=10$ feet

Hence $R-r=\frac{10}{2\pi}$ or $\frac{5}{\pi}$ or about 19 inches.

This is a startling answer to most. It becomes more dramatic if the teacher has the pupils estimate approximate answers before calculations begin. These questions have been used for the purpose:

Can a thin sheet of paper be slipped between the earth and the wire? Could the textbook be slipped between the earth and the wire? Could you crawl between the earth and the wire?

The next problem related the same problem to a basketball and ran as follows: Assume a basketball is a sphere of 38 inches circumference. Place a wire around the "equator" of the basketball. Cut the wire, add 10 inches of wire, and then place the wire so that it is everywhere equidistant from the equator of the basketball. How far is the wire from the basketball? The sharp students conquer this problem in a few minutes although they may be somewhat skeptical of the answer. It is most interesting to observe a class while its members are thinking about this second problem.

The third problem was similar to the first two. The sphere in this case was a baseball 9 inches in circumference. By this time most of the pupils had abstracted the generalization.

[2] C. Butler and L. Wren, *The Teaching of Secondary Mathematics* (New York: McGraw-Hill, 1941), pp. 119–128.

[3] L. B. Kinney and C. R. Purdy, *Leading Mathematics in the Secondary Schools* (New York: Rinehart, 1952), chapter 16.

They were now ready for the same problem with a sphere of circumference x units with y units of wire added.

The fourth problem can be stated as follows: assume the earth is a sphere of 25,000 miles circumference and that you are 5 feet tall. If you walked around the earth at the equator, how much farther would the top of your head travel than your feet (in a linear direction)?

Finally some two weeks following the initial problem the following recreation was given the class: Again assume the earth a sphere is of 25,000 miles circumference. A railroad circles the earth at the equator. The track is laid on a grade 3 feet above the ground. How long will the new railroad line be? The set of problems can be extended. Other possible learnings than those indicated may become evident.

This series of problems with the unexpected and yet provable answers is interesting to the pupils and obviously does far more than just amuse them. Other recreations were very popular and were useful teaching aids. Space does not permit incorporating them in this article.

In experimenting with the use of recreations, the members of the seminar found that: (1) Recreations are most useful when brought in at the appropriate time and in a context suitable for their purpose. Generally, the less highly organized the recreational activity and the more closely related to the regular work of the class, the more effective it becomes. (2) Care must be taken that the teacher never allows a class to feel that she is being "smart" or "showing off." (3) After a short period of time, the teacher should offer an explanation of the recreation. (4) The teacher must be "sold" and thoroughly understand the recreations she uses. There is nothing quite as sad as a mixed-up and unenthusiastically presented recreation.

One unexpected problem that perplexed the seminar for several weeks was a system for organizing the recreations in a form suitable for ready use in all the situations indicated previously. After several attempts involving notebooks a plan was devised that proved very effective. After the group had selected several hundred recreations appropriate for use in arithmetic and algebra, a nearby trade school duplicated the recreations on cards (about $3'' \times 5''$). These recreations were separated into three groups—arithmetic, algebra, and a "general teacher" group—and numbered. This last category contained all those recreations where the effect of the recreation would be spoiled by placing the copy in the hands of the pupil, or where reading to the class was the appropriate method of presentation. In this latter group only one copy was placed in each set. Two and three copies of the other recreations were provided in each set. Plywood boxes ($3'' \times 5'' \times 10''$) were made by industrial art classes to hold the cards. A very brief answer and explanation sheet was compiled. Over one hundred sets of the recreations were distributed to the junior high schools of the city.

Very little formal research has been done on the use of recreations. Three formal studies are worth reporting here. Porter determined the effect of mathematical recreations on the learnings in algebra and geometry.[4] A control technique was used. The experimental group was given a series of recreations closely related to the regular classwork during one period each week. The control group was given the usual work. On the final examination the experimental group made significantly superior grades. The additional learnings were ascribed to both the more favorable attitudes of the students toward mathematics and the superiority of recreations for teaching certain concepts. Brandes performed a similar experiment with general mathematics classes and obtained very similar results.[5] Hall used mathematical recreations as one means of redirecting the emotional outlook of college freshmen who were blocked in their thinking with regard to mathematics.[6] Results indicated that their use,

[4] R. E. Porter, "The Effect of Recreation in the Teaching of Mathematics," *School Review* (June 1938), pp. 423–27.

[5] Brandes, *op. cit.*, p. 327.

[6] A. J. Hall, "A Study of Non-Achievers in Mathematics," Unpublished Master's Thesis, Stanford University, 1941.

accompanied by an effort to relate mathematics to the student's goals, might prove a valuable approach to resolving the student's unfavorable outlook on the subject.

Is there any valid reason why students in the junior high school who generally take great delight in simple puzzles should be deprived of a similar pleasure when they study mathematics? The members of the seminar answered with a loud "NO." To capitalize on interest in what is novel and in winning a game, traits possessed by most students, is good teaching. Why don't you try mathematical recreations as a means of creating and maintaining interest and at the same time present many basic mathematical concepts dramatically?

What's new?

BOOKS

SECONDARY

Plane Geometry, C. R. Wylie, Jr., New York, McGraw-Hill Book Company, Inc., 1955. Cloth, viii+381 pp., $4.00.

Successful Devices in Teaching Algebra, edited by Katharine E. O'Brien, Portland, Maine, J. Weston Walch, Publisher, 1955. Paper, 150 pp., $2.50.

Successful Devices in Teaching Geometry, edited by Katharine E. O'Brien, Portland, Maine, J. Weston Walch, Publisher, 1955. Paper, 166 pp., $2.50.

Using Mathematics, Kenneth B. Henderson and Robert E. Pingry, New York, McGraw-Hill Book Company, Inc., 1955. Cloth, xvi+555 pp., $3.36.

COLLEGE

Analytic Geometry, New Second Edition, Ross R. Middlemiss, New York, McGraw-Hill Book Company, Inc., 1955. Cloth, ix+310 pp., $3.75.

Basic Mathematics for General Education, Second Edition, H. C. Trimble, L. C. Peck, F. C. Bolser, New York, Prentice Hall, Inc., 1948, 1950, 1955. Cloth, xiv+363 pp., $5.00.

College Algebra and Plane Trigonometry, Abraham Spitzbart and Ross H. Bardell, Cambridge 42, Massachusetts, Addison-Wesley Publishing Company, Inc. Cloth, xiii+408 pp., $4.50.

College Algebra (Revised edition), Paul R. Rider, New York, The Macmillan Company, 1955. Cloth, xiv+397 pp., $4.00.

Differential and Integral Calculus, Second Edition, Harold Maile Bacon, New York, McGraw-Hill Book Company, Inc., 1955. Cloth, vii+547 pp., $6.00.

Differential Equations, Frederick H. Steen, Boston, Massachusetts, Ginn and Company, 1955. Cloth, vii+330 pp., $4.00.

First Course in Algebra for Colleges, L. J. Adams, New York, Henry Holt and Company, 1955. Cloth, vi+217 pp., $3.00.

Higher Arithmetic, Virgil S. Mallory and Kenneth C. Skeen, Chicago, Illinois, Benjamin H. Sanborn and Company, 1955. Cloth, v+418 pp., $3.20.

Histoire abrégée des Sciences Mathématiques, René Dugas, Paris, France, Librairie Vuibert, 63, Boulevard Saint-Germain. Paper, 405 pp., Fr. 1350.

Integers and Theory of Numbers, Abraham A. Fraenkel, New York, Scripta Mathematica, 1955. Cloth, 102 pp., $2.75.

Introduction à la Méthode Statistique, Albert Monjallon, Paris, France, Librairie Vuibert, 63, Boulevard Saint-Germain, 1954. Paper, 280 pp., Fr. 2000.

Introduction to College Mathematics, Revised Edition, M. A. Hill and J. Burton Linker, New York, Henry Holt and Company, 1938, 1955. Cloth, xiv+529 pp., $5.25.

Modern Trigonometry, John C. Brixey and Richard V. Andree, New York, Henry Holt and Company, 1955. Cloth, xii+209 pp., $3.25.

Numerical Methods, Andrew D. Booth, New York, Academic Press, Inc., 1955. Cloth, vii+195 pp., $6.00.

Plane Algebraic Curves, E. J. F. Primrose, New York, St. Martin's Press, Inc., 1955. Cloth, vii+111 pp., $3.00.

Principles of Mathematics, C. B. Allendoerfer and C. O. Oakley, New York, McGraw-Hill Book Company, Inc., 1955. Cloth, xv+448 pp., $5.00.

The Continuum, Second Edition, Edward V. Huntington, New York, Dover Publications, Inc., 1917, 1955. vii+82 pp., Paper-bound: $1.00; Cloth-bound: $2.75.

Theory of Functions of a Real Variable, Leo F. Boron and Edwin Hewitt, New York, Frederick Ungar Publishing Company, 1955. Cloth, 277 pp., $6.50.

Theory of Functions, Volume 2, C. Caratheodory, translated by F. Steinhardt, New York, Chelsea Publishing Company, 1954. Cloth, 220 pp., $4.50.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

The magic of compound interest

The astounding rapidity with which a series of numbers in geometric progression ($r>1$) increases apparently never ceases to intrigue the neophyte. Witness the many pastimes based on this principle. The unbelievable phenomenon of tearing a single page of newspaper into two sections 22 times to make a pile higher than the Woolworth Building. Or asking a youngster to deposit 1¢ in a toy bank today, 2¢ tomorrow, 4¢ the next day, etc., for one month, not skipping Sundays. It will not be easy for him to believe that he will have to deposit $10,737,418.24 on the thirty-first day. Let alone that he only had to put in $5,368,709.12 the day before!

Then there is the classic Chinese legend of the chessboard problem: "If one grain of wheat is placed on the first square of a chess board, two grains on the second square, four on the next, then eight, sixteen, thirty-two, etc., how many grains will there be on the last, or sixty-fourth square?" To which the venerable sage replied that there was not enough wheat in all of China to supply enough grains for the last square alone, to say nothing of the entire chess board! As a matter of fact, this answer doesn't even begin to tell the story. For if the entire wheat crop of the world had been the same every year since the beginning of the Christian era as it has been in recent years, all the wheat produced for the last 2,000 years would still not be enough! There is also that other Oriental problem, the famous Tower of Hanoi, which, according to legend, required the priests to transfer 64 golden discs of different sizes from one spindle to another, one at a time, so that no disc should ever be smaller than those above it. Even with the help of a third empty spindle, the number of transfers required is $2^{64}-1$, or 18,446,744,073,709,551,615. According to Kasner and Newman,[1] allowing one transfer every second, 24 hours a day, year in and year out, it would require a little more than 58,454,204,609 *centuries* to complete the task!

One more pastime of this sort must suffice. It was recorded in the pages of THE MATHEMATICS TEACHER not long ago by Vera Sanford, and refers to "The Great Bet, and When It Will Be Paid." This oddity first appeared in Smith's *Practical and Mental Arithmetic*.[2] It concerns an election bet, based on the geometric progression 1, 2, 4, 8, . . . to 43 terms, which netted the winner $87,960,930,222.07; a sum which, if counted out at the rate of $60 per minute, would require over 9,789 years to complete!

One can argue that such examples, since the common ratio is 2, rather grossly exaggerate the "power of compound interest." But let us see.

When money is earning compound interest, the principal is multiplied, each conversion interval, by a number—not doubled, to be sure, but multiplied by a factor like 1.03 or 1.05. This may not seem like much of an increase, but it is enough to cause a sum of money to double itself in from 15 to 20 years at ordinary rates of interest. A well-known bank, in one of its advertisements some years ago, described an old Roman toy bank, about 1900 years old. It then went on to say

[1] Edward Kasner and James Newman, *Mathematics and the Imagination* (New York: Simon and Schuster, 1940), page 171.

[2] Roswell C. Smith, *Practical and Mental Arithmetic* (Hartford, Connecticut, 1838), p. 274.

that if one cent had been kept at 3 per cent interest compounded once a year from then until now, a period of 1900 years, it would have accumulated to $24,000,000,000,000,000,000,000. No wonder we speak of the magic of compound interest!

Here is one where the rate of interest is 4 per cent. It might have been called "Another Futile Wager," and was discovered in an old 1940 *World Almanac*. It seems that:

> Two friends made a wager in Baton Rouge, La., that will pay the winner $2,084,495,605.22, but it won't do either of them any good since the payoff will be in 2432. J. D. Stotler bet R. E. Collins that Louisiana's $5,000,000 capitol building will stand up 500 years. Collins bet $2.50 that it would not. They signed the papers and banked the money under contract in 1932. The two met in Baton Rouge in the middle of July, 1938, reaffirmed the bet and learned that the stake had grown to $6.85 from the 4 per cent interest rate that will boost it above two billions in 500 years. The contract specifies for the bank to pay the money to the heirs of the winner.

But the illustration to end all illustrations was given, nearly a hundred and fifty years ago, by one Francis Baily:[3]

> If a *penny* had been put out at 5 per cent interest at the birth of Christ; to what sum would it amount at the end of the year 1810? In that time, it would amount to .004166666 $\times(1.05)^{1810}$, which equals 938469,000000,000000,-000000,000000,000000 pounds. Now the diameter of the earth is about 8000 miles; consequently its solid contents will be 68,188963,498145,-531559,936000 cubic inches: and if it were made of standard gold, each cubic inch being worth £38 10*s*., the total value of such a globe would be 2625,275094,678602,965057,536000 pounds. But the amount of a penny in 1810 years, as above stated, is more than 357,474,600 times the value of such a globe: consequently, if one penny had been put out at 5 per cent compound interest at the time above mentioned, it would at this period have amounted to more money than could be expressed by THREE HUNDRED AND FIFTY-SEVEN MILLION of globes each equal to the earth in magnitude but of solid gold!!! Whereas, if it had been put out at the same rate of simple interest, the amount in the same time would have been only *seven shillings and sevenpence half-penny*.

In conclusion, the reader may find the following table of interest:

[3] Francis Baily, *The Doctrine of Interest and Annuities* (London, 1808), p. 96.

NUMBER OF YEARS IN WHICH A GIVEN AMOUNT WILL DOUBLE ITSELF AT INTEREST

Rate	*At Simple Interest*	*At Compound Interest, Compounded Quarterly*
1	100.00 yrs.	69.237 yrs.
1½	66.67	46.297
2	50.00	34.743
2½	40.00	27.748
3	33.33	23.191
3½	28.57	19.890
4	25.00	17.415
4½	22.22	15.490
5	20.00	13.949
5½	18.18	12.689
6	16.67	11.639
8	12.50	8.751
10	10.00	7.018
12	8.34	5.862

The great advance of mathematics

As E. T. Bell has pointed out: "There are good grounds for the frequent assertion that the nineteenth century alone contributed about five times as much to mathematics as had all preceding history. This applies not only to quantity, but, what is of incomparably greater importance, to power."[4]

In this connection it is most interesting to note the observations of A. N. Whitehead on the great advance of mathematics, as recorded by Lucien Price:[5]

> . . . as we were returning from dinner to the library, and when we were companionably settled there, a question was put which had long been pending:
>
> "Why has science kept advancing with such strides since 1900 when so much else has retrograded?"
>
> "One reason," said he, "is the great advance of mathematics between 1700 and 1900. Men of science have thus had a delicate and precise instrument with which to explore their new worlds."
>
> "But why such an advance in the past two centuries, when mathematics had been highly developed by the Greeks at least twenty-six centuries ago?"
>
> "Man's earlier discoveries in mathematics were made by observation of his physical sur-

[4] E. T. Bell, *The Development of Mathematics* (New York: McGraw-Hill, 1940), p. 15.

[5] Lucien Price, *The Dialogues of Alfred North Whitehead* (Boston: Little, Brown and Company, 1954), pp. 68–69.

roundings, as contradistinguished from abstract reason," said he. "On the plains of Chaldea he noticed the stars swinging round and round, deduced the conception of the circle, and finally arrived at the wheel. Now the wheel is not such an obvious invention as it sounds. As late as the fifteenth century, when America was found by the Europeans, the wheel had not been discovered in this hemisphere. Geometry, 'earth-measurement,' γεωγραφία, was developed by the Egyptians from their need to redraw boundaries obliterated in annual inundations of the Nile.

"But between these earlier discoveries, deduced from physical experience, and the later ones, only to be arrived at by abstract reason, occurred a long gap. Roman numerals were clumsy, and the Arabic numerals, easier to manage, did not reach Europe until the twelfth century. When they did, their simpler patterns, more readily seized by the eye, laid mathematics within the grasp of a wider variety of minds. By the end of the seventeenth century, this advance begun at the Italian Renaissance, culminated in Newton and Leibniz—logarithms, trigonometry, and algebra were developed, and the period closed with, if not the invention, certainly the perfecting of the integral calculus. The road was now open, from 1700 onward, for that excursion into applied mathematics which has equipped scientists with a complete and sensitive medium for creating thought-patterns with which to interpret their explorations of sensory phenomena."

"Even when allowance has been made for the historical vicissitudes, the collapse of the Roman Empire, the Dark Ages, and so on . . . it still seems odd that that long lapse should have occurred after such a hopeful start in the ancient world," I remarked.

"You get a lot of hopeful starts," said he, "and then one or another will lead through. It will take two hundred years to follow out into all their ramifications the leads which scientists have started. This can be done by really rather second-rate men, men with nippy minds who can follow assigned procedures within a limited field. But they are not original minds. Their achievement may wear the look of originality yet be so limited that it may not represent one one-thousandth of experience. Science has reached a point where it can transmit this facility of investigation; but it is of secondary worth. It takes no Shakespeare to do."

Some famous rejoinders

Men of worldly affairs often rise to an occasion with a brilliant rejoinder or a keen witticism. Mathematicians, despite the ivory tower in which the rest of mankind supposes them to be secluded, have likewise been known to retort with a lively *bon mot*. History records a number of instances of spontaneous remarks by famous mathematicians—at least pithy, if not always profound.

Legend has it that when a youthful student, who had begun to read geometry with Euclid and had finished the first proposition, inquired of the master: "What do I get by learning those things?" the renowned geometer called his slave and said: "Give him a piece of silver, since he must gain out of what he learns." In much the same way, when the great Alexander asked the mathematician Menaechmus to teach him geometry concisely, the latter is said to have replied: "O King, through the country there are royal roads and roads for common citizens, but in geometry there is but one road for all."

During the trying days of the French Revolution, when the famous French mathematician Lagrange, who laid the mathematical foundations of classical mechanics in his epoch-making treatise *Mécanique Analytique*, was called before the tribunal and asked what useful thing he could do to deserve life, he answered simply: "I will teach arithmetic." Probably few replies have ever been more sincere or more significant than this one.

Another brilliant mathematician, a contemporary of Lagrange, was the socially-minded Laplace, author of the *Mécanique Céleste*, a work which for the first time in history gave a complete mathematical exposition of the workings of the solar system. When Laplace, in the grand manner, offered Napoleon a copy of his book, Napoleon, who delighted in embarrassing people, said to the great mathematician: "M. Laplace, they tell me you have written this large book on the system of the universe, and have never even mentioned its Creator." Laplace, although usually artful and diplomatic, was nevertheless somewhat sensitive when it came to matters of science and philosophy. He drew himself up and replied bluntly: "Je n'avais pas

besoin de cette hypothèse-là." Napoleon, amused at hearing that the savant did not feel the need of God, told this reply to Lagrange, one of his favorites, who tactfully observed that the "hypothesis of God was admirable; it explains many things."

Many anecdotes are told of the distinguished Swiss mathematician Leonhard Euler, who lived during the middle of the eighteenth century. Naturally a timid man, he intensely disliked public affairs. Yet several times he was called upon by royalty. When invited to the court of Frederick the Great in Berlin in 1741, the Queen of Prussia was surprised to find that the renowned scholar was so reticent and shy. Upon expressing her amazement, Euler naively replied, "Madame, it is because I come from a country where, when one speaks, one is hanged."

Then there is the story of Euler's visit to the Court of Catherine II at St. Petersburg. It happened that the French philosopher Diderot, who was also at the Court, had been discussing atheism rather freely among the courtiers. Diderot was told that there was a mathematician present who could prove the existence of God algebraically; would he like to hear it? Diderot assented. Whereupon Euler reverently approached Diderot, and, in tones of great certainty, said solemnly: "Monsieur, $(a+b)^n/n=x$, donc Dieu existe; respondez!" Since algebra was "Greek" to Diderot, he was completely flabbergasted; amid hilarious laughter, the philosopher withdrew in embarrassment and chagrin.

The celebrated composer of *Recreations in Mathematics and Natural Philosophy*, Jacques Ozanam, a genial soul, had little inclination for theological pursuits, as his family wished. Although devout, he was averse to disputations about points of faith. On this subject he used to say, "It is the business of the Sorbonne to discuss, of the Pope to decide, and *of a mathematician to go straight to heaven in a perpendicular line.*"

The French mathematician Louis Poinsot (about 1820) is celebrated for his original contributions in geometrical mechanics. We recall his remark on the definition of time. "If anyone asked me to define *time*," said Poinsot, "I should reply: Do you know what it is that you speak of? If he said Yes, I should say, Very well, let us talk about it. If he said No, I should say, Very well, let us talk about something else." This last rejoinder, incidentally, is neither trivial nor nonsensical. For it is very much the same approach that is taken by the modern mathematician with respect to a logical system such as geometry. In such a system some few terms must inevitably be left undefined, since they do not admit of logical definition. The word "point" is an example of such a term, for it cannot be rigorously defined; neither can the word "line." Thus we might paraphrase Poinsot: Do you know what points and lines are? If you do, let us talk about points and lines in geometry. If you do not know what points and lines are, then let us talk about something else.

The teacher had been explaining fractions to her class. When she had discussed the subject at length, wishing to see how much light had been shed, she inquired: "Now, Dick, which would you rather have, one apple or two halves?"

The little chap promptly replied: "Two halves."

"Oh, Dick," exclaimed the young woman, a little curious, "Why would you prefer two halves?"

"Because I could see if a worm was inside!"

Reviews and evaluations

Edited by Richard D. Crumley, University of South Carolina, Columbia, South Carolina and Roderick C. McLennan, Arlington Heights High School, Arlington Heights, Illinois

BOOKS

Algebra One, Rolland R. Smith and Francis G. Lankford, Jr., New York, World Book Company, 1955. Cloth, vi+410 pp., $2.80.

Algebra Two, Rolland R. Smith and Francis G. Lankford, Jr., New York, World Book Company, 1955. Cloth, vi+506 pp., $3.00.

These two books are designed as textbooks for courses in beginning algebra and second year algebra at the high school level. *Algebra One* has chapters that deal with: the introduction to algebra; solution of linear equations in one unknown, including literal equations; operations with signed numbers; operations with polynomials; solution of fairly complex linear equations in one unknown and the use of such equations in solving problems; graphs of data, formulas, and linear equations in two variables; solution of pairs of linear equations in two variables by graphing, substitution, and elimination by addition and subtraction; finding special products of binomials and factors of trinomials; operations with fractions, solution of equations containing fractions, and solution of fractional equations; operations with roots and radicals; solution of quadratic equations in one unknown by graphing, factoring, completing the square, and by formula; the use of ratio, proportion, and variation; and the solution of right triangles by the use of the tangent, sine, and cosine trigonometric ratios. At the end of each chapter there is a review section, a list of new words used in the chapter, and a special interest section describing puzzles or incidents in the history of algebra. Readiness exercises appear from time to time to review skills or clarify concepts prior to their use in a new topic. "Extra" exercises and topics are provided for better students. The book is printed in two colors—red and black.

The best characteristic of *Algebra One* is the authors' subscription to the principles of inductive learning. This goal as stated in the preface is commendable, although they do not always use an inductive development nor are all the inductive developments used successful. A good example of the inductive development is the section called "Numerical Approach to Problem Solving" (pp. 148, 149), which helps the student formulate the algebraic expressions and equation necessary for solving a problem by showing how to assume definite values for the unknown numbers and work through the problem. The inductive development of the quadratic formula (p. 340) is also good. Another desirable feature is the inclusion of readiness exercises as preparation for new topics. Most of these sections, called "Before You Begin," are good.

The most serious defect of *Algebra One* in the opinion of the reviewer is that the concepts of literal number, variable, and function are not developed in accord with modern mathematics. The result is that these concepts are not developed as clearly as possible. For example, literal numbers are identified with "general numbers" (p. 2) which are developed intuitively as "any" numbers. Letters are introduced as symbols for these general numbers. Yet, on page 26, letters are used to represent a definite, unknown number without making any distinction between these two uses of letter symbols. The concept of variable is related to the concept of literal number on page 177 by the definition: "Literal numbers whose values may change are called *variables*." This confusion could have been avoided by a more careful treatment of a letter symbol as representing any number in a particular set of numbers. The concept of function as such is not developed anywhere in *Algebra One*. The reviewer could not find the word "function" used once. The *idea* of function does occur in the sections on variation.

Several of the definitions are questionable and many are descriptions rather than definitions. An example of a questionable definition is found on page 70: "A number having either a plus sign or a minus before it is called a *signed number*." This seems to the reviewer to confuse matters. A number is an idea, so how could any number have a plus sign or a minus sign before it? Even if the distinction between numbers and symbols were cleared up, this definition does not contribute much to the concept of signed numbers.

The use of two colors in printing makes an attractive book, but the extra color is used functionally in only one situation—to distinguish between "+" and "−" used as parts of the symbolic representation of positive and negative numbers and the four signs of operations. The illustrations and charts are well done; page layout is generally good; and only two typographical errors were found (p. 16 and p. 213).

In summary, *Algebra One* succeeds fairly well with the inductive type of development and is

adaptable for different students and for varying emphases in algebra courses. Some of the definitions and explanations will be objected to by many teachers. The content is traditional (although a bit more inclusive than many other textbooks). All in all the reviewer rates this book as average.

Algebra Two is designed for a course in second-year algebra. It has chapters that deal with: a review of the fundamental operations applied to signed numbers and polynomials; a review of the solution of linear equations and their use in solving problems; review and extension of special products and factoring; review and extension of fractions; solution of equations with fractions, including proportions; the coordinate system and graphing of linear equations; solution of systems of linear equations and their use in solving problems; variables and functions, and graphing of functions; operations with irrational and imaginary numbers; solution of quadratic equations in one unknown; quadratic equations with two variables; exponents; logarithms; the binomial theorem; arithmetic and geometric series; numerical trigonometry; and rates of change. At the end of each chapter is a review section and a special interest section. Extra exercises and topics are provided for better students. The book is printed in black and white.

As with *Algebra One*, the desirable inductive approach is used in *Algebra Two* except in the review sections. The best example of this approach is the chapter on logarithms, which is excellent. The feature of having extra exercises and topics is used effectively. The content is more inclusive than many other texts, making the book more adaptable. The charts and figures used are well done.

The major objections as seen by the reviewer are: (1) the lack of clear distinctions between the number symbols and the concepts of number and quantity; (2) the emphasis on developing skills rather than on developing understandings; (3) lack of adequate motivation, especially for factoring; (4) a few poorly worded definitions and sentences; and (5) an inexcusable number of typographical errors (pp. 91, 95, 103, 127, 204, and 250). An example of a poorly worded definition is: "A *sequence* is a succession of numbers that obey some fixed law" (p. 413). All real numbers obey the fixed law: $2x+3x=5x$, so according to the definition any succession of real numbers would be a sequence.

Weighing all these factors, the reviewer would rate *Algebra Two* as slightly above average.—*Richard D. Crumley*

Mathematics—A Third Course, Myron Rosskopf, Harold D. Aten, William Reeve, New York, McGraw-Hill Book Co., Inc., 1955. Cloth, iii+438 pp. $3.48.

It is pleasing to encounter a book which presents solid mathematics, not watered down for the benefit of students who cannot or will not study. This text covers algebra, a considerable amount of trigonometry, and some elementary analytic geometry. Some features are above average, for example the treatment of computation with approximate numbers is detailed, and, in general, better than usually found. The discussion of zero as a significant digit (p. 190) is fine. The authors ask for *an* equation which is represented by a table of values of x and y, rather than *the* equation which would imply there is a unique equation (it might have been worth while pointing this out to the student). They do point out why they ask the student to form *a* quadratic equation with given roots, rather than *the* quadratic equation. In introducing the mil as an angular measure they avoid the common error of defining this as the angle subtended by one yard at a distance of 1000 yards, and correctly state that this is an approximation. The distinction between sequences and series is carefully made. These are not the only good features, others will appeal to another teacher just as these appealed to this reviewer.

Some of the points to which this reviewer takes objection are found in other texts; it seems, however, that the number in this book is unreasonably large. In considering points which should be mentioned a reviewer must be careful to distinguish between treatments which he feels could be improved and statements which are clearly erroneous. The first case is a matter of opinion; in the second the situation is markedly different.

Under the first category there are many points. In several places the student is given a rule without a reason. Certainly the student who wonders "Why?" should find some answer in his text. The solution of a system of linear equations by third-order determinants is given as a mechanical rule, with no justification; it is not necessary to carry the relative error of measurement out to more than two significant digits (no reason given); rules are given for simplification of radicals with no reason other than that of saving labor in computations; a rule for finding equations of the asymptotes of a hyperbola has no explanation.

It is doubtful if the student will understand a statement such as "An asymptote is a line that is tangent to a curve at infinity." If a function repeats its values in any interval of length p, the period of the function is said to be p, usually we want the smallest period (p. 307); on page 318 we find that the period of a function is observed to be 4π—why is it *the* period rather than the smallest period? On page 286 we discover that for an angle to be in standard position its terminal side must fall in one of the four quadrants; on page 290 we are asked to put an angle of 90° in standard position—undefined for a quadrantal angle (if the angle has its vertex at the origin and its initial side along the positive x-axis, why state anything about the terminal side in defining standard position?). Many teachers object to the graphing of trigonometric functions with π at 3 units on the horizontal

axis. There are other definitions for the principal values of the inverse trigonometric functions. These might be mentioned. Even though the angles involved are acute, it would not be impossible to use directed lines, rather than ignore direction (p. 337).

Figure 39 spoils an otherwise excellent treatment of irrational numbers, for if $\sqrt{2}=1.414$ then $\sqrt{2}$ is a rational number. When we read that the precision of a measurement refers to the size of the unit in terms of which the measurement is expressed does the student have a right to assume that a girl's height expressed as 5.5 feet has different precision than if expressed as 65 inches? One doubts the complete accuracy of the statement "Since 81 is a perfect square, its square root is 9" (p. 61). On page 56 we find that $a+b$ and $a-b$ are *the* factors of a^2-b^2; are there no others? Many prefer to speak of the limit of the sum of an infinite geometric progression, rather than the sum.

The preceding points may be debatable, other errors are more serious. On page 118 we find that it is *customary* to write a radical in simplest form, on page 120 we find the result *must* be in such a form that the denominator does not contain a radical. Consistency is of value, and who issues the edict which says something *must* be done? Why has 24°10′ four significant digits (p. 191) but 23°10′ three significant digits (p. 268)?

The restriction that the base must not equal zero is omitted on page 111 and 114, but given in another situation (p. 113)—this seems inexcusable. " . . . If either the bases or the exponents are different, you must raise to a power first, and then perform those combinations that are possible" (p. 109). What about 2^3+6^3—*must* one raise to a power first?

"The principles developed for . . . division with radicals apply to imaginary numbers" (p. 140), yet on page 119, $\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab}$ with no restriction on a and b. The student who believes what the authors say is in for trouble.

Apparently (p. 24), the equation of any straight line can be written in the form $y=mx+b$; comment seems needless. In discussing trigonometric identities we are told that the reciprocal relationships hold for a general angle as well as an acute angle (false for certain functions of quadrantal angles); then to make matters worse an attempt is made on page 330 to claim that the identity $\tan\theta=\sin\theta/\cos\theta$ is true for $\theta=90°$, by means of a limiting process. The student often fails to distinguish between the impossible operation of a non-zero constant divided by zero, and the limit of a constant divided by x, as x approaches zero; when it is implied, as here, that these are equivalent it certainly merits violent objection. The statement is merely absurd—reference to standard texts would have shown that an identity is true for all values of the variable for which the expressions involved have meaning. On page 309, it is stated that as an angle approaches 90° its tangent increases without bound, without restriction as to how the angle approaches 90°. The student is asked to find additional terms of sequences such as 3, 6, 9, 12, . . . *Any* three terms can be given, which are correct, and a formula can be written covering the situation.

Every quadratic equation has two roots—no statement is made about multiple roots (a similar confusion exists on p. 168). The definition of a fractional exponent $a^{m/n}=\sqrt[n]{a^m}$ will produce confusion if a is negative and n even.

Even though teachers may feel some of these points are unimportant, a student is entitled to have a text which gives correct, rather than erroneous statements. There are texts available which in this matter are superior. It is indeed doubtful if the good features outweigh the bad ones in this case.—*Cecil B. Read, University of Wichita, Wichita, Kansas*

DEVICE

Blackboard Compass, The Speed-Up Geometry Ruler Company, Inc., 5100 Windsor Mill Road, Baltimore 7, Maryland. Wood, metal, and rubber device for drawing circles and arcs on chalkboard; 20″ long; $2.50 each in lots of 1 or 2, prepaid; $2.25 each in lots of 3 to 5, prepaid; $2.15 each in lots of 6 or more, prepaid.

This device is made with a 20″ section of a wood yardstick to which is attached a metal bracket at each end. One bracket holds a rubber suction cup, and the other a metal rod rounded at the end. A metal sliding bracket with a rubber bushing provides support for the chalk. The section of yardstick is printed in the conventional inch scale with $\frac{1}{8}$″ subdivisions. The sliding bracket which holds the chalk can be moved according to the radius desired up to 18″. To draw a circle, one places the suction cup on the chalkboard at the point where the center of the circle is to be and grasping the chalk held by the bracket rotates the device around the suction cup. The rounded metal stub held rigidly by the bracket at the opposite end of the stick slides on the chalkboard and maintains the stick in a position parallel to the chalkboard. When short arcs are desired, the device is turned over and a small rubber stub is used instead of the rubber suction cup. Replacement parts for the rubber pieces on the device are made available by the producer.

The device is well constructed and, with proper care, should be serviceable for a number of years. It is convenient for drawing fairly large circles which may be approximately centered vertically on the chalkboard. Since the metal stub at the extreme end transcribes a circle of radius almost 20″ each time a circle is drawn, one must allow enough clearance or the metal stub will bump the frame of the chalkboard. Another slightly objectionable feature is that the metal stub makes a tiny scratch on slate chalkboards. On a newer type chalkboard, no scratch was made. For most purposes, however, the device functions well.—*Richard D. Crumley*

• WHAT IS GOING ON IN YOUR SCHOOL?

Edited by John A. Brown, University of Wisconsin, Madison, Wisconsin and Houston T. Karnes, Louisiana State University, Baton Rouge, Louisiana

Originality in arithmetic

by William E. Beeman, North Texas Laboratory School, Denton, Texas

Any teacher of seventh-grade arithmetic is always trying to find some way to stimulate the pupils' interest as well as aid his learning of the subject. Last fall I was very pleased with the reaction of one seventh-grade section, in particular the work of one student, when some of the classic word-digit problems were presented. This student became so interested and spent so much time working on the problems that I asked her to write a paper about her experience. Her paper in her own words follows.

We started our discussion with

```
 SEND
 MORE
------
MONEY
```

All the other examples in her paper are her own.

A New Kind of Arithmetic

Until a few weeks ago we had a regular mathematics class. One day our teacher found a new kind of mathematics problem.

It goes like this, you take a problem made of words and change them to numbers. Be sure it all adds up right. The only catch is all the letters that are the same in the words have to be the same number after you change the problem.

Sample:

```
 BIG   756
 EGG   266
------ ----
TREE  1022
```

See how all the *g*'s are the same. The *e*'s are too. But remember not to use the same number for both *g*'s and *e*'s.

Here are some problems for you to work for yourself.

```
 SEND     STOP     STOP
 MORE     SPOT     LOOK
------   ------   ------
MONEY    NANCY    BILLY

 THEY     FINE     MOST
 EVEN     TOYS     EYES
------   ------   ------
ELOPE    TODAY    SHINE
```

Make up some of your own, too. But remember that just not every problem will work. If there are more than ten different letters in the problem to begin with, it will not work.

```
 ANNIE    RIGHT
 HURTS    SIDES
------   ------
PEOPLE   CLOSED
```

Can you see why these will not work?

Had you completely eliminated the possibility that there could be a problem like this?

```
 STOP
 SPOT
-----
SARAH
```

The only way this problem will work is to use *O*'s for the *S*'s and have a remainder from *T* plus *P*.

Subtraction is possible too. It works just like addition but you subtract. Sample:

```
 THEY
 FLEW
------
 HERE
```

Find the quotient. It works just like addition and subtraction.

GET
A) GIFT

Multiply. This is much harder than anything you will have tried thus far.

CUT
A
TREE

I sincerely hope you get as much fun out of word mathematics as I have.

Beverly Brooks,
Seventh Grade,
North Texas Laboratory School
Denton, Texas

The Janesville, Wisconsin, mathematics program

by Joseph Kennedy, Wisconsin High School, Madison, Wisconsin

The Janesville, Wisconsin, Junior-Senior High School has an enrollment of 1525 students in grades 9–12. Of these, 1045 are enrolled in mathematics courses! This is most noteworthy in a day of reported decreases in mathematics enrollments. Here are their enrollments by grades and courses:

Grade	*Courses and Enrollments*						*Total*
9	algebra	57%	math I	43%			100%
10	geometry	36%	math II	41%			77%
11	algebra	15%	math III	20%	arithmetic III	15%	50%
12	adv. math	10%	math IV	15%	refresher math	16%	41%

Mathematics I, II, III are recognized by the University of Wisconsin as equivalent to algebra I and geometry for college entrance.

Arithmetic III is for those people who did or did not take any mathematics in tenth grade, but scored low on a computation test given to all tenth-graders. Therefore, they were required to take this course in the eleventh grade for one semester, three days per week, with no outside preparation and carrying part credit.

Mathematics IV ends the general mathematics course which parallels the college mathematics course. Those who have had algebra II may take either advance mathematics or mathematics IV F (mathematics of finance). Mathematics IV M is the continuation of the second track course.

Refresher mathematics is for those who did not take any work in mathematics after the ninth or tenth grades. These students are encouraged to brush up before going out into the business world.

It is interesting to note that 64 per cent of all Janesville students in the upper three grades are enrolled in science courses. Both the physics and chemistry courses are run on a two track program.

Special credit for the interest in mathematics in Janesville Junior-Senior High School is due to the fine teaching staff headed by Marjorie Davis in the senior high and Ruth Jones in the junior high. This sound program has been developed through their efforts with the cooperation and interest in this work on the part of the administrators of the Janesville school system.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Sixteenth Christmas Meeting

Hotel Sheraton-Park, Washington, D. C.
December 27–30, 1955

The National Council of Teachers of Mathematics will hold its Sixteenth Christmas Meeting in our national capital, December 27–30, 1955, at the Sheraton-Park Hotel, where we have obtained excellent convention rates. The program and the special events planned for the meeting will be of interest to teachers of mathematics and supervisors of mathematics programs.

Tuesday, December 27, 1955. For the early arrivals the committee has arranged to have members of The National Council of Teachers of Mathematics and their families attend one of several teas given by the foreign embassies in Washington. This should prove to be an outstanding educational treat for members of the Council and their families. The number who can take advantage of this exceptional opportunity is limited, so reservations must be in the hands of the local arrangements committee at an early date.

Wednesday, December 28, 1955. This is the first day of the convention meetings. The program will open with an address by Dr. Raymond J. Seeger, Assistant Director of the National Science Foundation. Dr. Seeger will speak on "The Mathematical Sciences and the Problem of Manpower." In view of the serious manpower shortage in the physical sciences this address should prove to be of special interest. What can the teacher of mathematics do to alleviate this shortage?

Following Dr. Seeger's address there will be several sectional meetings devoted to elementary, junior high school, senior high school, and college mathematics. Every effort has been made by the committee on program arrangements to make these sectional meetings especially helpful to the classroom teachers. This part of the program will also include conferences and laboratories.

Thursday, December 29, 1955. The second day of the convention will continue in much the same pattern as that followed during the first day. Events of special interest to conventioneers are: (1) "The Youth Forum," conducted by Dr. Robert C. Yates, Virginia Polytechnic Institute, Blacksburg, Va. (Here the teen-agers will tell the group why they think high school students should study mathematics); (2) TV demonstration on the junior high school level; (3) a demonstration lesson in arithmetic by Miss Evelyn Ball, Washington, D. C. schools.

Dr. P. S. Jones, University of Michigan, Ann Arbor, Michigan, will speak at the afternoon general session on "The Continuity and Sequence of Mathematical Concepts—Grades One to Twelve."

The banquet is the high light of any National Council convention. On Thursday

evening Dr. H. P. Ettlinger, Chairman of the Department of Mathematics at the University of Texas, Austin, Texas, will deliver the banquet address on the topic "Mathematics as a Profession." Dr. Ettlinger is well known for his enthusiastic efforts to interest students in the high schools of Texas to continue in their study of mathematics. His message will be not only enjoyed by members of the Council, but it may be taken back to their classrooms to their students.

Friday, December 30, 1955. Any trip to Washington, D. C., should be climaxed by a trip to the White House. The local committee has made special arrangements to have a limited number of conventioneers visit the residence of our President. Watch your convention program for special announcements.

General comments. Special tours have been arranged to visit the Bureau of Standards, U. S. Naval Observatory, Capitol Building, and other points of interest. Arrangements can be made to provide for additional tours for families of members of the Council.

A program providing complete details will be mailed to members from the Executive Offices of the National Council.

Minutes of the Sixth Delegate Assembly

The National Council of Teachers of Mathematics
Statler Hotel, Boston, Massachusetts
April 14, 1955

Edited by H. Glenn Ayre, Chairman, Affiliated Groups,
Western Illinois State College, Macomb, Illinois

The Sixth Delegate Assembly of the National Council of Teachers of Mathematics met in the ballroom of the Statler Hotel in Boston, Massachusetts, at 8:00 A.M., Tuesday, April 14, 1955, with H. Glenn Ayre presiding.

Marie S. Wilcox, the president of The National Council of Teachers of Mathematics, brought greetings and expressed the importance of the Affiliated Groups in helping the teachers to know of the assistance which the National Council can give them—especially through their publications. She said these publications are not as well known as they should be and those who attend these National Council meetings should tell the others about them.

Glenn Ayre reported that of the 64 affiliated groups, 49 planned to send representatives. The Regional Representatives were introduced, and they, in turn, introduced the delegates from their districts. Dr. Ayre presented a map showing the geographical re-districting of the affiliated groups into six regions. This new division provides a more equitable distribution of NCTM members and local affiliated groups. It is hoped that this new division of territory will make for closer cooperation among the groups and their regional representatives and also make more efficient the participation of affiliated groups in selecting prospective nominees for officers and board members of NCTM.

The request of the Greater Toledo Council of Teachers of Mathematics to become affiliated with the National Council was officially granted by the Board, and this new member of our group was welcomed to the affiliated groups as the only new member this year. This brings the total number to 65.

Mr. Henry Swain, editor of the Handbook, presented the *Handbook for Organizations of Mathematics Teachers* and gave a short history of the project which was initiated two years ago by our former chairman, Mary Rogers. Mr. Swain took over the job of editor of the Handbook when Dr. Ayre assumed the chairmanship of the affiliated groups. The original Handbook Committee was H. G. Ayre, editor, Joyce Benbrook, Barbara Betts, Howard I. Gallant, Jeannette Garrett, Lucy A. Hall, Catherine A. V. Lyons, Lucille Martin, and Hubert Risinger. John R. Mayor was later added and asked to contribute a section.

Mr. Swain gave credit to all those who had

contributed to the project and spoke of the work of the editor in rewriting the material sent him by the committee so it would work into a unified document. It seems that Mr. Swain, Mrs. Swain, the Swain children, and a number of students from New Trier Township High School spent many hours mimeographing and assembling the Handbook.

Mr. Swain read the table of contents of the Handbook to give some indication of the wide variety of helps this guide has for new groups and even well-established groups who wish to improve their services.

Mr. Swain stated, "This first edition is in mimeographed form because there will be more ideas coming in and we want you to help to improve this by sending in suggestions for improvement."

Each affiliated group will receive one copy free and additional copies are available through the Washington Office at 50 cents each.

The meeting was then turned over to Dr. John Mayor, the moderator for the panel that discussed the values of affiliation. Dr. Mayor first asked Elizabeth Roudebush to speak on the need for places to meet, especially for the Christmas and summer meetings. Miss Roudebush mentioned the fact that nothing is better to get one's group together than having The National Council of Teachers of Mathematics meet with them. She asked that anyone who can act as host for the summer or the Christmas meetings let the National Council Board know of it.

The first panel speaker was Frank Allen, who spoke on the "Benefits to Affiliated Groups from the National Council." He spoke of the advantages as being: 1) Obtaining national leaders for local meetings; for instance, in Illinois five meetings were held, at which national leaders gave inspiring programs because they felt that Illinois was part of the team and they wanted to go and help them out. 2) Providing motivations for local leaders to work. 3) Giving each individual member a voice in national policies. 4) Giving the local group a chance to study national procedures and policies through newsletters, constitutions, and other publications.

Dr. Ellen Marth substituted for Dr. Margaret Willerding, who was stranded in New York because of the inclement weather. Dr. Willerding shared with Frank Allen the topic of "Benefits to Affiliated Groups from the National Council." The paper Dr. Willerding had prepared for the program pointed out, among other things, the following benefits to the affiliated groups from NCTM: 1) Affiliation broadens the scope of interest and promotes contact with problems on a national basis. 2) Local groups receive direction for their work by meeting requirements and standards established by the national organization. 3) To local groups just getting organized, it is especially helpful to be able to profit from the pioneering done by other groups. For example, the Missouri group was organized two years ago. The contacts with other Affiliated Groups and especially the newsletters from other organizations did much to make that organization and newsletter possible.

The opportunity for a local group to be host to a national meeting is one of the most stimulating and worth-while projects that a local group could undertake.

The next speaker was Mary Rogers, who talked on the "Benefits to The National Council of Teachers of Mathematics from the Affiliated Groups." She said there were about a half-dozen points, but she spoke on only one—membership. Last year The National Council of Teachers of Mathematics set their goal as 10,000. Last May they reached 10,001. The new goal is 15,000 to be reached by April, 1956. A membership committee is working with her. This committee will not solicit members but will serve as a clearinghouse, and each delegate is requested to write them informal letters or reports of their own membership plans. Miss Rogers told of the new membership plan whereby for $3.00 individual membership fee, one may choose to receive either *The Mathematics Teacher* or *The Arithmetic Teacher*, the latter of which will have six issues per year instead of four.

Mary spoke of the membership goals which her committee had set up for each state. Nevada had already exceeded their goal by three. She also suggested that we continue the newsletters, and give publicity to The National Council of Teachers of Mathematics at local meetings, making clear that there is given to National Council members a special price on publications important to mathematics teachers. Mary suggested that our slogan should be: "Every member win one." At the conclusion of her talk, Dr. Mayor said "the goal will soon be 20,000 if Mary has her way."

Harold Fawcett gave a very inspirational talk. He stated the strength of The National Council of Teachers of Mathematics does not lie in the Council, but in the individuals. He said you only need to get one-half a member—then they become whole. By strengthening the parent body, one uncovers latent talent. Twenty thousand members will not mean anything if they are not improved teachers.

A question was then raised as to whether one had to take out two memberships if he wanted both magazines, *The Mathematics Teacher* and *The Arithmetic Teacher*. Mr. Ahrendt replied, "Beginning with the October issue, $3.00 will bring either magazine and $2.00 more will get you the other."

Mr. Ahrendt then introduced the State Representatives. He gave as their functions: 1) Sell the Council to prospective members. 2) Announce new regulations, such as those on nominations. 3) Get publicity for The National Council of Teachers of Mathematics through local newsletters. 4) Gather information for The National Council of Teachers of Mathematics. He spoke of the wide variation in select-

ing State Representatives, but, now with the growth of the affiliated groups, the present practice is to get the local group to appoint a representative. It was believed that a committee is studying the possibility of combining the affiliated groups with the state representatives.

William Gager then spoke on the mixed-up situation existing in group affiliation—some small local groups and others large state groups. Quite a discussion followed this talk on how affiliation should be made. The Chicago group that holds Charter Number I does not wish to relinquish it, even for progress' sake. Others felt that more interest could be secured through small groups than through large groups, while others felt that if too many small groups affiliated, the organization might become too unwieldy, especially in voting, as well as resulting in too much duplication in membership and affiliation.

The discussion pointed out among other things that these organizations should meet the needs of the teachers, capture their enthusiasm, and channel their efforts toward better local accomplishments and a stronger National Council. No change should be made that would discourage the creation of local organizations or the affiliation with NCTM.

Hubert Risinger gave a report on newsletters and the Traveling Exhibit. He introduced Miss Messner, who told where the exhibit has been and where it is going. She made a plea for new materials.

Hubert then asked the question, "Newsletters, what to do about them?" This topic was discussed at great length. A motion was finally made by Miss Malter and seconded by Mrs. Kelly of New Jersey, that we have an affiliated Newsletter with about three or four issues a year—free to the officers of each affiliated group, but with provisions made for others to secure them for a nominal fee, if they so desire. This motion was unanimously passed. Dr. Ayre mentioned the fact that charges for this Newsletter will have to be taken up with the Executive Board.

The meeting, which was intensely interesting and worthwhile, adjourned at 11:10 A.M., instead of 10:30 A.M. as planned.

Respectively submitted,
Helen Malter,
Ella Marth,
Secretaries of Delegate Assembly

The following people were chosen by their local groups as official delegates and alternates to the Delegate Assembly. The name listed first is that of the delegate. If two names are listed, the second one is that of the alternate. Although some were unable to be present, the attendance was good and the affiliated groups were well represented.

Arkansas Council of Teachers of Mathematics
Lyle Dixon, State College, Jonesboro
Mary Lee Foster, Henderson State Teachers College, Arkadelphia

California Mathematics Council
William H. Glenn, Jr., South Pasadena
Marian C. Cliffe, Glendale

Colorado Council of Teachers of Mathematics
Mary Doremus, Denver

Delaware Council of Mathematics Teachers
Russell Dineen, Wilmington

Benjamin Banneker Mathematics Club
Jonelle B. Washington, Washington, D. C.

The District of Columbia Teachers of Mathematics
Faith Novinger, Anacostia Senior High School, Anacostia
Ella Marth, Wilson Teachers College, Washington, D. C.

Florida Council of Teachers of Mathematics
Kenneth P. Kidd, University of Florida, Gainesville
W. A. Gager, University of Florida, Gainesville

Dade County Council of Teachers of Mathematics
Eva H. Hoke, Miami

Hillsborough County Mathematics Council
Gladys Bonner, Franklin Junior High School, Tampa
Agnes Jeffries, Brandon High School, Tampa

Pinellas County Council of Teachers of Mathematics
Martha Harper, St. Petersburg
Alma Hayman, St. Petersburg

Mathematics Panel—State of Georgia
Rufus B. Godwin, Atlanta
Elizabeth R. Collier, Forsyth

Illinois Council of Teachers of Mathematics
Frank Allen, Lyons Township High School and Junior College, La Grange
H. Glenn Ayre, Western Illinois State College, Macomb

Chicago Elementary Teachers Mathematics Club
Joseph J. Urbancek, Chicago Teachers College, Chicago
Ramona H. Goldblatt, Chicago

Men's Mathematics Club of Chicago and Metropolitan Area
Ralph Briggs, Bloom Twp. High School, Chicago Heights
E. H. C. Hildebrandt, Northwestern University, Evanston

Women's Mathematics Club of Chicago and Vicinity
Martha Hildebrandt, Proviso Township High School, Proviso

Indiana Council of Teachers of Mathematics
Olive G. Wear, Fort Wayne
Philip Peak, Indiana University, Bloomington

Gary Council of Teachers of Mathematics
Izelia McWilliams, Gary

Iowa Association of Mathematics Teachers
Paul Shaw, Stone Wallace Jr. High School, Waterloo

Wichita Mathematics Association
C. B. Read, University of Wichita, Wichita
Lester Weinstein, Mathewson Intermediate School

Louisiana-Mississippi Branch of The National

Council of Teachers of Mathematics
Z. L. Loflin, Southwest Louisiana Institute, Lafayette
Houston T. Karnes, Louisiana State University, Baton Rouge

Prince Georges County Teachers Association—Mathematics Section
Walter Ortel, Bladensburg Junior High School, Bladensburg

Detroit Mathematics Club
Alfred Capoferi, St. Clair Shores

Michigan Council of Teachers of Mathematics
Russell Schneider, Eastern High School, Lansing
Nickoline A. Bye. Mt. Pleasant

Missouri Affiliated Group of the National Council of Teachers of Mathematics
Margaret Willerding, Harris Teachers College, St. Louis

Nebraska Section—National Council of Teachers of Mathematics
Milton Beckman, University of Nebraska, Lincoln
Theodora Nelson, Nebraska Teachers College, Kearney

The Association of Teachers of Mathematics in New England
Kenneth G. Fuller, New Britain, Connecticut

Southern New England Preparatory School Mathematics Association
Roderick Beebe Jr., The Gunnery School, Washington, Connecticut
Tom Finley, Loomis School, Windsor, Connecticut

Association of Mathematics Teachers of New Jersey
May J. Kelly, Longport
Ernest Renucci, Fanwood

Association of Mathematics Teachers of New York State
Myron F. Rosskopf, Columbia University, New York
Elaine Rapp, Oceanside High School, Oceanside

Nassau County Mathematics Teachers Association
Charles R. Salit, Hampstead
Florence Elder, West Hampstead High School, Hampstead

Suffolk County Mathematics Teachers Association
Edna Van Wort, Simpson High School, Huntington
Charles Kissam, Northport High School, Northport

Association of Teachers of Mathematics of New York City
Saul Landau, James Monroe High School, The Bronx
Harry Ruderman, Bronx High School, New York

Department of Mathematics of the North Carolina Education Association
Annie John Williams, Durham

Ohio Council of Teachers of Mathematics
Emmalou Brumfield, Kent State University, Kent
Eugene Smith, University School, Ohio State University, Columbus

Mathematics Club of Greater Cincinnati
Marie Backer, Cincinnati
Norma Stuke, Cincinnati

Greater Cleveland Mathematics Club
Doris Kreuger, John Marshall High School, Cleveland
Albine Weitzel, Euclid High School, Euclid

Tulsa Mathematics Council
Carolyn Stewart, Edison Junior High School, Tulsa
Alpha Savenius, Wilson Junior High School, Tulsa

Ontario Association of Teachers of Mathematics and Physics
Jim Kerr, Leaside Collegiate Institute, Leaside, Ontario

Oregon Council of Teachers of Mathematics
Lesta Hoel, Public Schools, Portland

Pennsylvania Council of Teachers of Mathematics
Edward Bosman, Stroudsburg
Lee Boyer, State College, Millersville

Western Pennsylvania Association of Teachers of Mathematics
Helen Malter, Pittsburgh
Mable Love Baker, Rosedale, Verona

Mathematics Section, East Tennessee Education Association
Ruth Benson, Oak Ridge High School, Oak Ridge

Texas Council of Teachers of Mathematics
Lois Averitt, Box 102, Sanger
Pearl Bond, Beaumont

Greater Dallas Mathematics Association
Lorena Holder, Dallas 8

Houston Council of Teachers of Mathematics
M. S. Keathley, Bellaire
Irene Furman, Houston

Utah Council of Teachers of Mathematics
Orla Bell, Salt Lake City
Eva Crangle, Salt Lake City

Mathematics Section, Virginia Education Association
Gladys Schuder, Charlottesville

Richmond Branch of the National Council of Teachers of Mathematics
Jeanette Henna, Thomas Jefferson High School, Richmond
Mary C. T. Edwards, Westhampton Junior High School, Richmond

Puget Sound Council of Teachers of Mathematics
Elizabeth Roudebush, Public Schools, Seattle

West Virginia Council of Mathematics Teachers
Jones L. Griffin, St. Albaus
Mrs. Robert Thomson, Beckley

Wisconsin Mathematics Council
Sister Mary Felice, Mount Mary College, Milwaukee
Miss Lydia R. Goerz, Kenosha

• NOTES FROM THE WASHINGTON OFFICE

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Yearly financial report

Below is given a summary of receipts and expenses of the Council for the fiscal year ending May 31, 1955. In the past, financial reports have been made only to the Board of Directors. The Board decided during the Thirty-third Annual Meeting, April 13–16, 1955, that yearly reports should also be made to the general membership.

Although the above report is largely self-explanatory, comments on some of the items may aid your understanding of it. The receipts from the sale of memberships and subscriptions are given according to the old membership provisions. In the future, with members having a choice of journals (*The Mathematics Teacher* or *The Arithmetic Teacher*), a new breakdown will be required.

It will be noted that our second largest source of income is the sale of publications. As would be expected, the best seller among the yearbooks during the past year was the latest yearbook, the twenty-second. Sixty per cent of the income from the sale of yearbooks came

Receipts and expenditures of The National Council of Teachers of Mathematics for the fiscal year, June 1, 1954—May 31, 1955

Balance forwarded, June 1, 1954	$32,702.45	
Receipts		
Membership and THE MATHEMATICS TEACHER subscriptions	$34,986.15	
The Arithmetic Teacher subscriptions	6,577.33	
Mathematics Student Journal subscriptions	5,669.26	
Sale of advertising space in *The Mathematics Teacher*	3,434.25	
Sale of advertising space in *The Arithmetic Teacher*	1,169.19	
Miscellaneous	376.65	
Net return from conventions	34.01	
Sale of publications		
Yearbooks	13,980.97	
Miscellaneous	8,476.02	
Total receipts		$ 74,703.83
		$107,406.28
Expenditures		
Washington Office	$24,227.11	
President's office	1,178.06	
The Mathematics Teacher	22,424.29	
The Arithmetic Teacher	4,474.06	
Mathematics Student Journal	3,233.71	
Contribution to NEA Building Fund	1,000.00	
Miscellaneous	2,130.19	
Printing and preparation of publications	6,870.86	
Total expenditures		$65,538.28
Balance, May 31, 1955		$41,868.00

from this one item. The best sellers among the miscellaneous publications were the five new booklets which were prepared by the Committee on Supplementary Publications and which came off the press during the months of January through April. Forty per cent of the income from the sale of the miscellaneous publications came from these five booklets.

A few words should probably be said about the expenditures. The expenditures charged to the Washington Office include all costs for salaries (for a staff of five persons); office forms and supplies; mailing (except the journals) of publications, renewal notices, and letters; membership, promotion, and travel by the executive secretary. Expenditures charged to the president's office include, in addition to travel and office expenses, payments from the Special Projects Fund. The figure for each journal includes the total expense for printing, mailing, and editorial costs. The contribution to the NEA Building Fund was voted by the Board of Directors as an expression of appreciation for the many services that the NEA gives without charge to the Council. The miscellaneous expenses include travel to the Annual Meeting by the Board of Directors, committee work, and certain special authorized expenses.

Our rather good cash balance and the fact that it increased during the year indicate that the Council is in sound financial condition. Not all of our bank balance, however, is expendable. A significant part of our cash balance consists of funds that have been paid us by our subscribers for future work on our journals. An amount sufficient to cover these obligations must be held in reserve. Funds should also be reserved for the printing of future publications, especially yearbooks, three of which are now in preparation. One reason for the increase in cash resources during the year was that no yearbook was published during the past fiscal year.

Readers who have further questions are invited to write to the executive secretary at the Washington office.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, The National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

CHRISTMAS MEETING

December 27–29, 1955
Sheraton-Park Hotel, Washington, D. C.
Veryl Schult, local chairman, District of Columbia Teachers College, Washington 9, D. C.

ANNUAL MEETING

April 11–14, 1956
Schroeder Hotel, Milwaukee, Wisconsin
Margaret Joseph, 1504 N. Prospect Avenue, Milwaukee 2, Wisconsin

JOINT MEETING WITH NEA

July 2, 1956
Portland, Oregon
Lesta Hoel, Public Schools, Portland, Oregon

SUMMER MEETING

August 19–22, 1956
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

Other Professional Dates

Chicago Elementary Teachers' Mathematics Club
December 12, 1955
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Anne T. Linehan, O'Toole School, Chicago, Illinois

A column of unofficial comment

National Council news

by Marie S. Wilcox, President, The National Council of Teachers of Mathematics

The National Council of Teachers of Mathematics has for several years held four meetings each year: the annual meeting in the spring, a summer meeting, a Christmas meeting, and a one-day meeting at the time of the meeting of the NEA.

Last year two committees recommended that the Board of Directors consider reducing this number of meetings. These committees were the Convention Committee, which had been appointed by the Board to make suggestions for the improvement of convention routine, and the Place of Meeting Committee. Also the Committee on Affiliated Groups made a suggestion that regional meetings be considered. It is the desire of the Board to arrange meetings so as to be satisfactory to as large a group of the membership as is possible. At the August meeting of the Board of Directors the matter of the number of meetings was referred to the Convention Committee and the Committee on Affiliated Groups for further consideration and recommendation. Organizations affiliated with the National Council are urged to discuss the matter of meetings and make suggestions or recommendations to their delegates, who will attend the Delegate Assembly next April. They are also urged to communicate directly with the chairman of the Convention Committee, Miss Irene Sauble, 16503 Glastenbury Rd., Detroit 19, Michigan, and the chairman of the Committee on Affiliated Groups, Dr. H. Glenn Ayre, Western Illinois State College, Macomb, Illinois. Some questions for study are:

1. How many meetings of national scope should the National Council sponsor each year?
2. Should some meetings be eliminated and replaced by regional meetings?
3. If we have regional meetings, should they be scheduled at the same time in all parts of the country? Would Christmas and/or summer be appropriate times for regional meetings?
4. Would regional meetings tend to compete unfavorably with state and local meetings?

Some definite action will probably be taken on this matter at the meeting of the Board at the time of the annual meeting of the Council next April in Milwaukee. The cooperation of all members of the Council in finding a satisfactory answer to this problem is solicited. If you are not a member of a group affiliated with the National Council, you are welcome to send suggestions about meetings to the chairmen mentioned above.

Many National Council committees were appointed in 1954 for a two-year period to coincide with the term of the President. However, some standing committees are organized in such a way that a certain number of members retire each year. Also, during the past year, a few committee members have resigned. Some of these resignations have been made in order to accept other Council assignments.

A list of the committees appointed in 1954 appears in the December, 1954, issue

of THE MATHEMATICS TEACHER. New appointments are listed below. Appointments are for a period ending at the close of the annual meeting of the year indicated.

Executive

Phillip S. Jones, Ann Arbor, Michigan (1956)
H. Vernon Price, Iowa City, Iowa (1956)

Supplementary Publications

Henry Swain, Winnetka, Illinois, Chairman (1958)
Marguerite Brydegaard, San Diego, California (1958)
Kenneth Kidd, Gainesville, Florida (1958)
Jesse Osborn, St. Louis, Missouri (1958)

National Council Business: Affiliated Groups

Northeastern: Catherine Lyons, Pittsburgh, Pennsylvania (1958)
Southeastern: Houston Banks, Nashville, Tennessee (1958)
Central: Adeline Riefling, St. Louis, Missouri (1957)

Auditing

Gwendolyn Holland, Washington, D. C. (1956)

Budget

John R. Mayor, Madison, Wisconsin, Chairman (1957)
Houston Karnes, Baton Rouge, Louisiana (1958)

Membership Drive

Nellie Alexander, Edwardsville, Illinois (1956)
Faith Novinger, Washington, D. C. (1956)

Relations with NEA

Allene Archer, Richmond, Virginia, Chairman (1957)
Josephine Berkeley, Washington, D. C. (1957)
Virginia Lee Pratt, Omaha, Nebraska (1957)

Nominations for Editor of THE MATHEMATICS TEACHER (1956)

Bruce Meserve, Montclair, New Jersey, Chairman
Charles Butler, Kalamazoo, Michigan
E. H. C. Hildebrandt, Evanston, Illinois
Lenore John, Chicago, Illinois
F. G. Lankford, Farmville, Virginia
Zeke Loflin, Lafayette, Louisiana

Place of Meeting

Forest N. Fisch, Greeley, Colorado (1958)
Hubert Risinger, East Orange, New Jersey (1958)

Publicity Subcommittee (Popular Magazines)

Barbara Betts, Boston, Massachusetts, Chairman (1956)

Research Committee

Harold Trimble, Cedar Falls, Iowa, Chairman (1956)
Nathan Lazar, Columbus, Ohio (1958)

Nominations and Elections (1956)

Maurice Hartung, Chicago, Illinois, Chairman
Milton Beckman, Lincoln, Nebraska
Clifford Bell, Los Angeles, California
Ida May Bernhard, Austin, Texas
F. G. Lankford, Farmville, Virginia
John R. Mayor, Madison, Wisconsin
R. R. Smith, Springfield, Massachusetts
Ben Sueltz, Cortland, New York
F. Lynwood Wren, Nashville, Tennessee

Curriculum Committee: Elementary

Fred J. Weaver, Boston, Massachusetts, Chairman (1958)
Joyce Benbrook, Houston, Texas (1957)
Ann Peters, Keene, New Hampshire (1956)

Irene Sauble, Detroit, Michigan (1958)
Esther Swenson, University, Alabama (1957)
Henry Van Engen, Cedar Falls, Iowa (1956)

Secondary

Frank B. Allen, La Grange, Illinois, Chairman (1958)
Clifford Bell, Los Angeles, California (1958)
Howard F. Fehr, New York, New York (1957)
John R. Mayor, Madison, Wisconsin (1957)
Bruce Meserve, Montclair, New Jersey (1956)
Sheldon Myers, Columbus, Ohio (1956)

Editors of the three Yearbooks now in process have all selected their planning committees with the approval of the Board of Directors. These committees are:

F. Lynwood Wren, Nashville, Tennessee, Chairman
Carl B. Allendoerfer, Seattle, Washington
Saunders MacLane, Chicago, Illinois
Bruce Meserve, Montclair, New Jersey
Carroll V. Newsom, Albany, New York

Phillip S. Jones, Ann Arbor, Michigan, Chairman
Harold Fawcett, Columbus, Ohio
Alice Hach, Racine, Wisconsin
Charlotte Junge, Detroit, Michigan
Henry Syer, Boston, Massachusetts
Henry Van Engen, Cedar Falls, Iowa

Maurice Hartung, Chicago, Illinois, Chairman
Frank Allen, La Grange, Illinois
Robert Pingery, Champaign, Illinois
Fred J. Weaver, Boston, Massachusetts
Donovan Johnson, Minneapolis, Minnesota

Two new topics for council publications have been approved. Editorial staffs for these books will be announced soon. The topics are: "The Joint Contribution of Mathematics and Other Fields Toward the Achievement of Our Common Educational Goals" and "The Teaching of Arithmetic."

The NCTM Research Award

At the time of the 1952 Christmas meeting of The National Council of Teachers of Mathematics, the Board of Directors approved a recommendation of the Research Committee of the Council that a research award be made in the form of a $1000 scholarship to be awarded to the individual who has a master's degree and who submits the best prospectus for a research study in learning problems in the field of mathematics, including arithmetic.

Notices of this action were sent to colleges and universities and an announcement was published in the October, 1953 issue of THE MATHEMATICS TEACHER. Certain basic conditions needed to be satisfied, and a prospectus was to be submitted before January 1, 1955. The award was to be made at the annual meeting of the Council in 1955 if a prospectus deemed worthy of being so honored was received.

Prior to January 1, 1955, a prospectus had been received from each of six people. The Board of Directors then appointed a panel of judges to review the material received and make recommendations concerning an award. The panel consisted of Zeke Loflin, Henry Van Engen, and Bruce Meserve, Chairman.

After careful consideration of the manuscripts submitted, the panel recommended that no award be made for 1955.

The committee made several suggestions with respect to other award possibilities. As a result of these suggestions, the Board of Directors requested the Research Committee to reconsider the matter of awards and the encouragement of research and make more specific recommendations to the Board at a later date.

News about future research awards will appear in this journal.

Fifteen years of classroom testing produced this new book . . .

SHOP MATHEMATICS

By CLAUDE E. STOUT

General Motors Institute, Flint, Michigan

Here is a book that has been developed and tested for fifteen years in mimeographed form at the General Motors Institute. Carefully revised every two years during this period, *Shop Mathematics* gives a clear presentation of arithmetic, algebra, geometry, trigonometry, logarithms, and the slide rule as they apply to actual shop practice.

Algebraic methods are provided in an easily understood manner. Throughout the text, emphasis is placed on checking results. No previous training in mathematics is necessary, and only a minimum of out-of-class preparation is required. All sections of the book are exceptionally well-integrated—methods explained in one section are used in subsequent sections. There is no unnecessary "topical" coverage to get in the way of fundamentals. The author holds chiefly to essentials and to shop applications. Arranged in four parts, the book lends itself to term teaching units of twelve weeks with two class hours per week.

1955 **282 pages** **Prob. $3.70**

Send today for your on-approval copy

JOHN WILEY & SONS, Inc. **440 Fourth Ave., New York 16, N.Y.**

NOW in its Second Printing

THE LEARNING OF MATHEMATICS

Its Theory and Practice

Twenty-First Yearbook of the National Council of Teachers of Mathematics

The wide acceptance of this yearbook is indicated by the fact that only a few copies remain of the large first printing.

Applies the most recent discoveries concerning the nature of the learning process to the problems of your mathematics classroom.

Discusses many questions about drill, transfer of training, problem-solving, concept formation, motivation, sensory learning, individual differences, and other problems.

Authorities consider it a significant contribution to the literature in mathematics education.

Price, $4.00. To members of the Council, $3.00.

Postpaid if you send remittance with your order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1201 Sixteenth Street, N.W. **Washington 6, D.C.**

A Constant Source of Inspiration in Teaching

HANDBOOKS TO AID MATHEMATICS

By SAMUEL I. JONES

Fascinating material of all types and kinds from junior high school thru the college level. Invaluable for club activities and supplemental classroom use. Extensively used and acclaimed the world over.

MATHEMATICAL WRINKLES

Today this volume is even more popular and just as valuable as when the following commendation was received a number of years ago.

"Kindly send us another dozen *Mathematical Wrinkles*. The Mathematics Department of Alexander Hamilton High School considers this book one of the best of its kind ever published, particularly well adapted to recreational mathematics for the high school student. At present, each member of our Mathematics Club, receives a copy of the work upon his graduation, not only as a reward for superior work done, but also to insure continued interest and enjoyment of a subject in which he has already evinced exceptional ability."—Ralph P. Bliss, Chairman, Department of Mathematics, Alexander Hamilton High School, Brooklyn, New York.

94 illustrations **376 pages** **5" x 7⅜"** **$3.50 each**

MATHEMATICAL NUTS

"I have a copy of each of your three books devoted to interesting problems in mathematics, the most interesting of which I think is *Mathematical Nuts*. Never before have I seen such a collection of problems that stagger one's admiration as these do."—E. P. S., Teacher, Francis T. Nicholls High School, New Orleans, Louisiana.

"I am delighted with the books and know I shall find them useful next year. Have always had nut-cracking every Friday in class, and I find this a most popular sport."—A. V., San Diego, California.

200 illustrations **352 pages** **5" x 7⅜"** **$3.50 each**

MATHEMATICAL CLUBS AND RECREATIONS

"*Mathematical Clubs and Recreations* contains numerous helps for clubs and recitation work together with numerous other recreations which the teacher can use with her pupil. Also offers suggestions as to what mathematical clubs can do in the way of keeping up interest and at the same time make a definite contribution to the mathematics instruction within the classroom."—The Bulletin, published by National Association of Secondary School Principals.

61 illustrations **256 pages** **5" x 7⅜"** **$3.00 each**

You are invited to write for copies on approval or for additional information.

Send orders to us or ask at your favorite bookstore.

S. I. JONES COMPANY, *Publisher*

1122 Belvidere Drive **Nashville 4, Tennessee**

PUBLISHERS OF BETTER BOOKS FOR BETTER TEACHING

TWO NEW SERIES featuring

— effective use of illustrations and color
— organization planned to aid teaching
— review of earlier work at all stages
— material for motivation closely related to the text

Essential Mathematics Series

Book 1 Arithmetic at Work
Book 2 Arithmetic in Life (in preparation)
by HOWARD F. FEHR and VERYL SCHULT
for Grades 7 and 8 and junior high school

Algebra, Course 1, Course 2

by HOWARD F. FEHR, WALTER H. CARNAHAN, MAX BEBERMAN
for Grade 9 and high school classes

For your bulletin board: send to the nearest Heath office for your free copy of ALGEBRA ON THE MAP, a poster of historical material from Fehr, Carnahan, Beberman

D. C. Heath and Company

SALES OFFICES: NEW YORK 14 CHICAGO 16 SAN FRANCISCO 5 ATLANTA 3 DALLAS 1
HOME OFFICE: BOSTON 16

Third printing just completed

NUMBER STORIES OF LONG AGO

by David Eugene Smith

A delightful illustrated account in story form of the probable history of numbers.

For both teachers and students.

Contains a section on number puzzles.

A classic that has been in demand for many years. 160 pages.

Price, 75¢ each. Quantity Discounts.

Postpaid if you send remittance with order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W.
Washington 6, D.C.

HOW TO DEVELOP A TEACHING GUIDE IN MATHEMATICS

By Mildred Keiffer and Anna Marie Evans

Tells how to develop teaching guides in mathematics to meet your local situation.

Discusses steps in the development of a guide, principles involved, content, and use.

Contains an annotated bibliography of reference materials.

No. 2 in the new "How To" series. All items in the series punched for three-ring binder for convenience in use.

10 pages. 40¢ each.

Please send remittance with order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W.
Washington 6, D.C.

Please mention the MATHEMATICS TEACHER when answering advertisements

ALGEBRA for Problem Solving

BOOKS ONE AND TWO

FREILICH • BERMAN • JOHNSON

A complete course in high school algebra that teaches for understanding as well as for use. With abundant exercises, problems, and tests; reviews and summaries for each chapter; and effective use of a second color as a teaching aid, this unusually sound program efficiently meets today's classroom needs.

MAKING MATHEMATICS WORK

NELSON • GRIME

This general mathematics text gives high school students a comprehensive review of arithmetic, reteaches fundamental processes, and introduces statistics, practical geometry, and algebra. Here are presented everyday problems for which skill in mathematics is essential including budgeting, buying on credit, and using a checking account.

BOSTON
NEW YORK
CHICAGO

HOUGHTON MIFFLIN COMPANY

DALLAS
ATLANTA
PALO ALTO

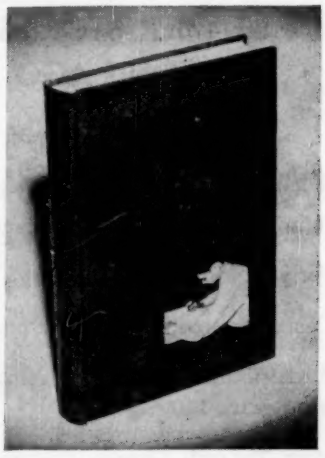
The Finest High School Mathematics Texts

MATHEMATICS FOR EVERYDAY LIVING — ***Leonhardy, Ely***

Mathematics applied to everyday problems, plus a thorough review of basic arithmetic operations.

ALGEBRA IN EASY STEPS — ***Stein***

First-year algebra geared for overcoming individual student difficulties. Almost 17,000 carefully graded exercises. Two editions—cloth and paper.



DYNAMIC PLANE GEOMETRY — ***Skolnik,***
DYNAMIC SOLID GEOMETRY — ***Hartley***

The concept of motion, stressing the visual approach, makes geometry more interesting and more meaningful.

PLANE TRIGONOMETRY, *2nd Edition* — ***Weeks, Funkhouser***

Three editions—without tables, with 4-place tables, or with 5-place tables. Carefully organized to allow mastery of one new topic at a time.

D. VAN NOSTRAND COMPANY, INC.

120 Alexander Street — **Princeton, New Jersey**

Don't Miss These Successful New Publications

RECREATIONAL MATHEMATICS, by W. L. Schaaf

This bibliography, probably the most comprehensive ever published, lists recreational materials in 58 categories under the following main headings: general works, arithmetical and algebraic recreations, geometric recreations, assorted recreations, magic squares, the Pythagorean relationship, famous problems of antiquity, and mathematical miscellanies. 136 pages. $1.20 each.

GEOMETRY GROWING, by W. R. Ransom

An interesting collection of materials. Presents "bits of geometry growing" from Pythagoras to Newton. Valuable for enrichment. Throws new light on many famous theorems. 40 pages. 75¢ each.

BYROADS OF ALGEBRA, by Margaret Joseph

Gives enrichment material on the level of high-school algebra. Discusses algebra to simplify arithmetic computations, number puzzles, and algebraic acrobatics. Useful for class or club. 16 pages. 40¢ each.

HOW TO STUDY MATHEMATICS, by Henry Swain

A popular seller, this handbook was prepared for high-school students and written in their language. Contains many practical suggestions for succeeding in homework, classwork, taking tests, and the like. Gives special suggestions for studying some of the difficult areas in high-school mathematics. Illustrated. 32 pages. 50¢ each.

HOW TO USE YOUR BULLETIN BOARD, by D. A. Johnson and C. E. Olander

Will give you a wealth of ideas for the effective use of your bulletin board. Discusses purposes, topics, supplies, techniques, and "tricks of the trade." Profusely illustrated. Will help you make a genuine learning tool of your bulletin board. Most of first printing already sold. 12 pages. 50¢ each.

Postpaid if you send remittance with order.
Quantity discounts.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W. Washington 6, D.C.

3 MODERN TEXTS

ARITHMETIC FOR TEACHER-TRAINING CLASSES

TAYLOR and MILLS

The new Fourth Edition of this well-known text is designed to insure a mastery of arithmetic processes and to give insight into the best methods of presenting them in the classroom.

MATHEMATICS FOR THE SECONDARY SCHOOL

W. D. REEVE

"Teachers of mathematics, those preparing prospective teachers . . . will find in this book an abundance of examples of good teaching procedures, ways of motivating the topics, of arousing and maintaining interest. . . . Few are as well qualified to advise them as the author."—E. R. BRESLICH, *University of Chicago*

MODERN TRIGONOMETRY

BRIXEY and ANDREE

A new text stressing trigonometric identities and equations, inequalities, and graphs of trigonometric functions. Less emphasis is on the numerical solution of triangles. Answers and skeleton solutions serve as a real teaching aid.

HENRY HOLT and CO.

New York 17 San Francisco 5

Over Eighty!

CLASSICAL TREATISES ON MATHEMATICS

have been translated into English, or reprinted in the original German or French, by CHELSEA.

World-famous treatises, texts, and monographs in Mathematics, Statistics, and Logic.

Send Today for FREE Catalogue

Chelsea, Dept. 2D, 552 W. 181 St., N.Y. 33, N.Y.

Please send me, without obligation, a copy of your FREE catalogue (#207) listing reprints and translations of Scientific Books.

Name .....

Address .....

City ..... Zone.....

State .....

MT November '55

Announcing a new algebra series—

ALGEBRA | ALGEBRA

FIRST COURSE | SECOND COURSE

by John R. Mayor & Marie S. Wilcox

A new and different approach to teaching algebra courses

Through its unique approach to teaching algebra, this new series meets the requirements of the modern secondary school curriculum. It offers pupils and teachers many distinct advantages, such as:

- makes the algebra course *enjoyable* for pupils and teachers
- helps the pupil learn by *reason,* not rule
- develops appreciation of algebra as a mathematical "whole"
- prepares pupils for later courses in mathematics
- provides for pupils of *every* aptitude

. . . and many others

Among the unique features in the Mayor and Wilcox texts are: a different order of presenting the traditional topics of algebra . . . highly effective method of developing algebraic concepts . . . material geared for pupils of *every* aptitude . . . a host of teaching and learning aids, including over 200 illustrations, exciting use of color, wide variety of problem material, review tests, vocabulary studies, cumulative reviews, and many others.

Written by outstanding authorities in the field, this new series will soon be available for your use. *Algebra, First Course* is scheduled for publication this January. *Algebra, Second Course* is expected to be ready for Fall 1956 classes.

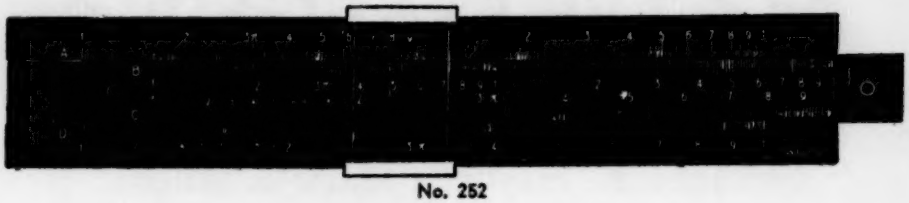
For further information on this new series, write the address below.

Educational Book Division

PRENTICE-HALL, Inc.

Englewood Cliffs New Jersey

The Large Welch Demonstration Slide—Rule



No. 252

4 FEET LONG

is now considered

ESSENTIAL EQUIPMENT

by most teachers of Mathematics, Physics and Chemistry.

NOT CUMBERSOME

Large—Clear—Scales and Numerals

Can be EASILY READ AT A DISTANCE

The Welch Demonstration Slide Rule operates Smoothly and has legible scales.

It simulates the small rules in every way. Results of computation performed with it are illustrative of those obtained with standard slide rules. The regular Mannheim A, B, C, and D scales are one meter long.

White scales and numerals on a black face provide greater legibility.

Two metal hangers are provided at the top edge so that the rule can be hung on the wall and rubber feet are provided to prevent marring the wall.

SUBSTANTIALLY CONSTRUCTED—FOUR LAMINATIONS OF MASONITE SECURELY RIVETED TOGETHER.

Each $13.50

Serving the schools for 75 years.

WRITE FOR CIRCULAR

W. M. WELCH SCIENTIFIC COMPANY

DIVISION OF W. M. WELCH MANUFACTURING COMPANY

Est. 1880

1515 Sedgwick St. Dept. X Chicago 10, Ill., U.S.A.